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Summation of threshold singularities in QCD

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Plan of Talk

- **Introduction**
- **Method: relativistic quasipotential approach for two arbitrary-mass particles**
- **Threshold resummation factor and its behavior**
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1. Introduction

As is well known, a description of quark-antiquark systems close to threshold does not permit us to cut off the perturbative series even if the expansion parameter, the QCD coupling α_s, is small

[T. Appelquist, H D. Politzer, Phys. Rev. Lett. 34 (1975) 43; Phys. Rev. D 12 (1975) 1404]. The problem is well known from QED [J. Schwinger, Particles, sources, and fields, Chap. 5-4, 1973].

The reason is that α_s is not the real expansion parameter in the threshold region – singular terms $(1/v)^n$ are also present.

$v = \sqrt{1 - \frac{4m^2}{s}}$ is a quark velocity, m is a quark mass.

The threshold singularities of the form $\left(\frac{\alpha_s}{v}\right)^n$ must be summarized.

In the nonrelativistic of case for the Coulomb interaction

$$V(r) = -\frac{\alpha}{r}\,, \tag{1}$$

such a resummation is realized the well-known Gamov–Sommerfeld–Sakharov S-factor [G. Gamov, Zeit. Phys. 51 (1928) 204; A. Sommerfeld, Atombau und Spektrallinien. Vieweg, v. II, 1939; A.D. Sakharov, Zh. Eksp. Teor. Fiz. 18 (1948) 631]

$$S_{\rm nr} = \frac{X_{\rm nr}}{1-\exp(-X_{\rm nr})}\,, \qquad X_{\rm nr} = \frac{\pi\,\alpha}{v_{\rm nr}}\,, \qquad (X_{\rm nr} = 2\pi\eta) \tag{2}$$

which is related to the wave function of the continuous spectrum at the origin via $|\psi(0)|^2$. For the case of higher ℓ states

$$L_{\rm nr} = \frac{X_{\rm nr}}{1-\exp\left[-X_{\rm nr}\right]} \prod_{n=1}^{\ell} \left[1 + \left(\frac{X_{\rm nr}}{2n}\right)^2\right]. \tag{3}$$

In the relativistic theory, the nonrelativistic approximation must be modified. The relativistic modification of the S-factor (2) in QCD in the case of two particles of equal masses ($m_1 = m_2 = m$) was performed in papers: V.S. Fadin, V.A. Khoze, Yad. Fiz. 48 (1988) 487; V.S. Fadin, V.A. Khoze, A.D. Martin, and A. Chapovsky, Phys. Rev. D 52 (1995) 1377. It consists in the substitution $v_{\rm nr} \to v$. This factor was used for the description of effects close to the threshold of pair production in $e^+e^- \to t\bar{t}$ and $e^+e^- \to W^+W^-$ processes.

Note, the substitution $\alpha \to 4\alpha_{\rm s}/3$ must be made is applied to QCD problems.

The same form of the S-factor as in above papers for the case of interaction between two particles of equal masses was proposed later in paper A.H. Hoang, Phys. Rev. D 56 (1997) 7276.

Another form of the relativistic generalization of the S-factor also in the case of two particles of equal masses was obtained in papers J.-H. Yoon and C.-Y. Wong, Phys. Rev. C 61 (2000) 044905; J. Phys. G: Nucl. Part. Phys. 31 (2005) 149.

The relativistic S-factor for two particles of arbitrary masses (m_1, m_2) was presented in paper A.B. Arbuzov, Nuov. Cim. A 107 (1994) 1263. This factor was derived within the relativistic quantum mechanics on the basis of the Schrödinger equation.

The new approach to contracting a relativistic generalization of the S-factor for the case of the interaction of two equal-mass particles was proposed by Milton and Solovtsov [K.A. Milton, I.L. Solovtsov, Mod. Phys. Lett. A 16 (2001) 2213]. It proved to be convenient to use, in this case, the relativistic quasipotential (RQP) of Logunov and Tavkhelidze [A.A. Logunov, A.N. Tavkhelidze, Nuov. Cim. A 29 (1963) 380] in the form proposed by Kadyshevsky [V.G. Kadyshevsky, Nucl. Phys. B 6 (1968) 125].

In Milton–Solovtsov paper, use was made of the transition of quasipotential (QP) equation from momentum space into relativistic configurational representation (RCR) introduced in the paper V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov, Nuov. Cim. A 55 (1968) 233 for the case of the interaction of two equal-mass particles. It is important to note that one has used the potential (1) which possesses the QCD-like behaviour [V.I. Savrin, N.B. Skachkov, Lett. Nuov. Cim. 29 (1980) 363].

For large Q^2 the potential $V \sim 1/(Q^2 \ln Q^2)$, which reproduces the principal behavior of the QCD potential proportional to $\bar{\alpha}_s(Q^2)/Q^2$ with $\bar{\alpha}_s(Q^2)$ being the QCD running coupling.

Thus, a new step in applying the quasipotential approach in QCD was made by Milton and Solovtsov. This approach leads to an expression for the relativistic S-factor in the form

$$S(\chi) = \frac{X(\chi)}{1 - \exp\left[-X(\chi)\right]}\,, \; X(\chi) = \frac{\pi\,\alpha}{\sinh\chi}\,, \; \sinh\chi = \frac{v}{\sqrt{1-v^2}}\,, \qquad (4)$$

where χ is the rapidity related to the total c. m. energy of interacting particles, $\sqrt{s}$ by $2m\cosh\chi = \sqrt{s}$.

The application of the relativistic S-factor in the form (4) to describing a number of features hadronic processes can be found in papers K.A. Milton, I.L. Solovtsov, O.P. Solovtsova, Phys. Rev. D 64 (2001) 016005; 65 (2002) 076009; Mod. Phys. Lett. A 21 (2006) 1355, where a new model expression for the Drell ratio $R(s)$, in which threshold singularities were summarized to the main potential contribution, was suggested.

- Ratio of hadronic to leptonic τ-decay widths in the vector channel

$$\boldsymbol{R}_\tau^V = R^{(0)} \int_0^{M_\tau^2} \frac{ds}{M_\tau^2} \left(1 - \frac{s}{M_\tau^2}\right)^2 \left(1 + \frac{2s}{M_\tau^2}\right) R(s)$$

- "Light" Adler function (constructed from τ-decay data)

$$\boldsymbol{D}(Q^2) = -Q^2 \frac{d\,\Pi(-Q^2)}{d\,Q^2} = Q^2 \int_0^\infty ds\, \frac{R(s)}{(s+Q^2)^2}$$

- Smeared function

$$\boldsymbol{R}_\Delta(s) = \frac{\Delta}{\pi} \int_0^\infty ds' \frac{R(s')}{(s-s')^2 + \Delta^2};$$

- Hadronic contribution to the anomalous magnetic moment of the muon

$$\boldsymbol{a}_\mu^{\text{had}} = \frac{1}{3} \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \frac{ds}{s} K(s)\, R(s)$$

and to the running of the fine structure constant

$$\boldsymbol{\Delta}\alpha^{(5)}_{\rm had}(s) = -\frac{\alpha(0)}{3\pi}\, s\,\mathcal{P}\int\limits_0^\infty \frac{ds'}{s'}\,\frac{R(s')}{s'-s}.$$

A common feature of all these quantities and functions is that they are defined via the function $R(s)$, the normalized hadronic cross-section, integrated with some other functions.

The suggested model [K.A. Milton, I.L. Solovtsov, O.P. Solovtsova, Mod. Phys. Lett. A 21 (2006) 1355] allows us to describe these quantities rather well.

The possibility of using the QP approach to define the relativistic S-factor is based on the fact that the BS amplitude, which parameterizes the physical quantity $R(s)$, is taken at $x = 0$, therefore, in particular, at relative time $\tau = 0$. The QP wave function is defined as the BS amplitude at $\tau = 0$, and the R-ratio can be expressed through the QP wave function $\psi_{\rm QP}(\mathbf{p})$ by using the relation

$$\chi_{\rm BS}(x=0) \,=\, \int d\Omega_p \, \psi_{\rm QP}(\mathbf{p})\,,$$

$d\Omega_p$ is the relativistic three-dimensional volume element in the Lobachevsky space realized on the hyperboloid $E_p^2 - \mathbf{p}^2 = m^2$.

The main objective of present study is to generalize the method proposed for getting (4) in order to derive the S/L-factor in the case of the interaction of two relativistic particles having unequal masses.

2. Method

- We begin our consideration by presenting the completely covariant relativistic-quasipotential equation written in the momentum space and constructed in paper [V.G. Kadyshevsky, M.D. Mateev, R.M. Mir-Kasimov, Yad. Fiz. 11 (1970) 692] for the relativistic-quasipotential wave function $\Psi_{q'}(\mathbf{p'})$ in the case of interaction between two relativistic particles that have unequal masse ($m_1\,, m_2$; in the following $c = \hbar = 1$):

$$\left(2E_{q'} - 2E_{p'}\right) \Psi_{q'}(\mathbf{p'}) = \frac{2\mu}{m'(2\pi)^3} \int d\Omega_{\,\mathbf{k'}} \widetilde{V}\left(\mathbf{p'}, \mathbf{k'}; E_{q'}\right) \Psi_{q'}(\mathbf{k'})\,, \qquad (5)$$

where

$$d\Omega_{\,\mathbf{k'}} = \frac{m^{'} d\mathbf{k'}}{E_{k'}}$$

is the relativistic three-dimensional volume element in the Lobachevsky space, $E_{k'} = \sqrt{m'^{\,2} + \mathbf{k'}^{\,2}}$, $m' = \sqrt{m_1\, m_2}$; $\mu = m_1 m_2/(m_1 + m_2)$ is the usual reduced mass.

Equation (5) represents a relativistic generalization of the Schrödinger equation in the spirit of Lobachevsky geometry which is realized on the upper half of the mass hyperboloid $E_{k'}^2 - \mathbf{k}'^{\,2} = m'^{\,2}$. This equation describes the scattering on the quasipotential $\widetilde{V}\left(\mathbf{p}', \mathbf{k}'; E_{q'}\right)$ of an effective relativistic particle having the mass m' and the relative 3-momentum $\mathbf{k}'$.

The quasipotential $\widetilde{V}\left(\mathbf{p}', \mathbf{k}'; E_{q'}\right)$ depends parametrically on the energy $E_{q'}$ of the effective relativistic particle. Thereby, the effective relativistic particle of mass m' plays the role of a two-body system and carries the total c.m. energy of interacting particles, $\sqrt{s}$, proportional to the energy $E_{k'}$ [V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov, Sov. J. Part. Nucl. 2 (1972) 69]:

$$\sqrt{s} = \sqrt{{m_1}^2 + \mathbf{k}^2} + \sqrt{{m_2}^2 + \mathbf{k}^2} = \frac{m'}{\mu}\sqrt{m'^2 + \mathbf{k}'^2} = \frac{m'}{\mu} E_{k'} \, . \qquad (6)$$

• The proper Lorentz transformation means a translation in the Lobachevsky space. The role of the plane waves corresponding to these translations are played by the following functions:

$$\xi(\mathbf{p}',\mathbf{r}) = \left(\frac{E_{p'} - \mathbf{p}'\cdot\mathbf{n}}{m'}\right)^{-1-ir\,m'}, \tag{7}$$

where the modal of the radius-vector, $\mathbf{r}$, ($\mathbf{r} = r\,\mathbf{n}\,, |\mathbf{n}| = 1$) is a relativistic invariant [V.G. Kadyshevsky, R.M. Mir-Kasimov, N B. Skachkov, Sov. J. Part. Nucl. 2 (1972) 69].

These functions correspond to the principal series of unitary representations of the Lorentz group and in the nonrelativistic limit ($p' \ll 1$, $r \gg 1$)

$$\xi(\mathbf{p}',\mathbf{r}) \to \exp(i\mathbf{p}'\cdot\mathbf{r}).$$

The functions (7) satisfy the equation in terms of finite differences

$$\left(2E_{p'} \,-\, \hat{H}_0\right)\,\xi(\mathbf{p'}\,,\mathbf{r}) = 0\,. \tag{8}$$

Here

$$\hat{H}_0 = 2\,m'\,\left[\cosh\left(i\lambda'\frac{\partial}{\partial r}\right) + \frac{i\lambda'}{r}\sinh\left(i\lambda'\frac{\partial}{\partial r}\right) - \right.$$
$$\left. -\,\frac{\lambda'^{\,2}\,\Delta_{\theta,\varphi}}{2r^2}\exp\left(i\lambda'\frac{\partial}{\partial r}\right)\right] \tag{9}$$

is the operator of the free Hamiltonian, while $\Delta_{\theta,\varphi}$ is its the angular part, and $\lambda' = 1/m'$ is the Compton wavelength associated with the effective particle of mass m'.

The wave functions in momentum space and in the r representation (it is known as the relativistic configuration representation, RCR) are related by the equation

$$\psi_{q'}(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d\Omega_{\mathbf{p}'}\, \xi(\mathbf{p}', \mathbf{r})\, \Psi_{q'}(\mathbf{p}')\,,$$
$$\Psi_{q'}(\mathbf{p}') = \int d\mathbf{r}\, \xi^*(\mathbf{p}', \mathbf{r})\, \psi_{q'}(\mathbf{r}). \tag{10}$$

For a spherically symmetric potential the application of transformations (10) to Eq. (5) leads to the equation which is the integral form of the relativistic Schrödinger equation in the RCR:

$$\frac{1}{(2\pi)^3} \int d\Omega_{\mathbf{p}}\, (2E_q - 2E_p)\, \xi(\mathbf{p}, \boldsymbol{\rho}) \int d\boldsymbol{\rho}' \xi^*(\mathbf{p}, \boldsymbol{\rho}')\, \psi_q(\boldsymbol{\rho}') =$$
$$= \frac{2\,\mu}{m'}\, V(\rho\,; E_q)\, \psi_q(\boldsymbol{\rho})\,, \tag{11}$$

where we introduced the new variables:

$$
\begin{gathered}
\mathbf{q}' = m'\mathbf{q}\,, \mathbf{p}' = m'\mathbf{p}\,, \mathbf{q} = \sinh(\chi_q)\mathbf{n}_q\,, \mathbf{p} = \sinh(\chi_p)\mathbf{n}_p\,, \\
|\mathbf{n}_q| = |\mathbf{n}_p| = 1\,, \boldsymbol{\rho} = m'\mathbf{r}\,, \boldsymbol{\rho}' = m'\mathbf{r}'\,, \rho = |\boldsymbol{\rho}|\,, \rho' = |\boldsymbol{\rho}'|\,, \\
r = |\mathbf{r}|\,, r' = |\mathbf{r}'|\,, d\mathbf{r}' = m'^{-3} d\boldsymbol{\rho}'\,, d\Omega_{\,\mathbf{p}'} = m'^{3} d\Omega_{\,\mathbf{p}}\,, \\
d\Omega_{\,\mathbf{p}} = \frac{d\mathbf{p}}{E_p}\,, E_{q'} = m'E_q\,, E_{p'} = m'E_p\,, E_q = \sqrt{1+\mathbf{q}^2}\,, \\
E_p = \sqrt{1+\mathbf{p}^2}\,, V(r\,; E_{q'}) = V(\rho/m'\,; E_{q'}) \equiv m'V(\rho\,; E_q)\,, \\
\xi(\mathbf{p}'\,, \mathbf{r}) = (E_p - \mathbf{p}\cdot\mathbf{n})^{-1-i\rho} \equiv \xi(\mathbf{p}\,, \boldsymbol{\rho})\,, \\
\psi_{q'}(\mathbf{r}) = \psi_{m'q}(\boldsymbol{\rho}/m') \equiv \psi_q(\boldsymbol{\rho})\,, \Psi_{q'}(\mathbf{p}') \equiv m'^{-3}\Psi_q(\mathbf{p})\,.
\end{gathered}
\tag{12}
$$

Here the right-hand side is already local in RCR and the transform of the quasipotential, $V(\rho\,; E_q)$, is given in terms of the same relativistic plane waves:

$$
V(\rho\,; E_q) = \frac{1}{(2\pi)^3}\int d\Omega_{\,\mathbf{p}}\,\xi(\mathbf{p}\,,\boldsymbol{\rho})\,\widetilde{V}\left(\mathbf{p}^2\,; E_q\right)\,.
$$

By using of the expansions

$$\xi(\mathbf{p}\,,\boldsymbol{\rho}) = \sum_{\ell=0}^{\infty}(2\ell+1)\,i^{\ell}\,p_{\ell}(\rho\,,\cosh\chi_{p})\,P_{\ell}\left(\frac{\mathbf{p}\cdot\boldsymbol{\rho}}{p\,\rho}\right), \qquad (13)$$

$$\psi_{q}(\boldsymbol{\rho}) = \sum_{\ell=0}^{\infty}(2\ell+1)\,i^{\ell}\,\frac{\varphi_{\ell}(\rho\,,\chi)}{\rho}\,P_{\ell}\left(\frac{\mathbf{q}\cdot\boldsymbol{\rho}}{q\,\rho}\right),$$

and also formula [V.G. Kadyshevsky, R.M. Mir-Kasimov, N.B. Skachkov, Nuov. Cim. A 55 (1968) 233]

$$p_{\ell}(\rho\,,\cosh\chi) = \frac{(-1)^{\ell}\,(\sinh\chi)^{\ell}}{\rho^{(\ell+1)}}\left(\frac{d}{d\cosh\chi}\right)^{\ell}\left(\frac{\sin\rho\chi}{\sinh\chi}\right),$$

Eq. (11) is transformed to the form

$$\frac{2}{\pi}\int_0^\infty d\chi' \frac{(\sinh\chi')^{2\ell+2}(-1)^{\ell+1}}{\rho^{(\ell+1)}}\left(2\cosh\chi - 2\cosh\chi'\right)\times$$
$$\times\left(\frac{d}{d\cosh\chi'}\right)^{\ell}\left(\frac{\sin\rho\chi'}{\sinh\chi'}\right)\left(\frac{d}{d\cosh\chi'}\right)^{\ell}\frac{1}{\sinh\chi'}\times \tag{14}$$
$$\times\int_0^\infty d\rho' \frac{\rho'\,\sin\rho'\chi'}{(-\rho')^{(\ell+1)}}\,\varphi_\ell(\rho',\chi) = \frac{2\,\mu}{m'}\,\frac{V(\rho\,;E_q)\,\varphi_\ell(\rho,\chi)}{\rho}\,.$$

$\chi'\,,\chi$ are the rapidities which are related to $E_p\,,E_q$ as $E_p = \cosh\chi'$, $E_q = \cosh\chi$, and the function

$$(-\rho)^{(\ell+1)} = i^{\ell+1}\frac{\Gamma(i\rho+\ell+1)}{\Gamma(i\rho)} \tag{15}$$

is the generalized power where $\Gamma(z)$ is the gamma-function.

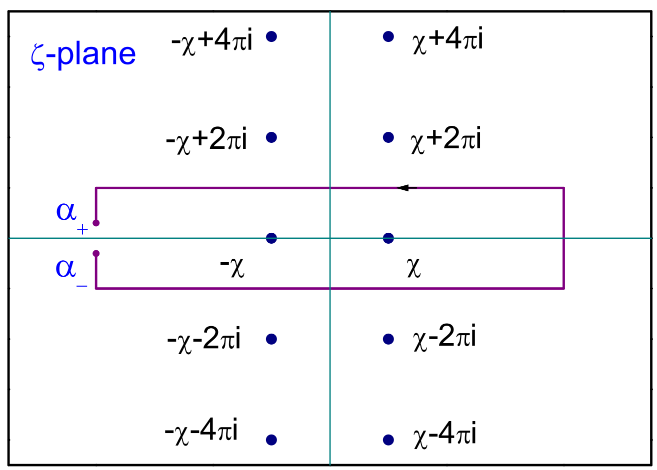
Thus, Eq. (14) differs from the corresponding equation in the case of two particles of equal masses only by the factor $2\mu/m'$ turning into 1 at $m_1 = m_2$ [I.L. Solovtsov, Yu.D. Chernichenko, Proc. of the Int. Seminar Denoted to the Memory of I. L. Solovtsov, Dubna, 15-18 Jan. 2008. JINR. Dubna. 2008. D4-2008-65, 73].

• To solve quasipotential Eq. (14), we seek a solution with the potential (1) in the form

$$\varphi_\ell(\rho\,,\chi) = \frac{(-\rho)^{(\ell+1)}}{\rho} \int\limits_{\alpha_-}^{\alpha_+} d\zeta\, e^{i\rho\zeta}\, R_\ell(\zeta\,,\chi)\,, \qquad (16)$$

where the ζ-integration is performed in the complex plane over a contour with end points α_- and α_+ (Fig. 1):

$$\alpha_- = -R - i\varepsilon\,, \alpha_+ = -R + i\varepsilon\,, R \to +\infty\,, \varepsilon \to +0.$$



- The resulting solution represented in terms of hypergeometrical function by

$$\varphi_0(\rho\,,\chi) = -N_0(\chi)(-\rho)^{(1)}\,e^{i\rho\chi+iA\chi}\,F\left(1-iA,1-i\rho;2;1-e^{-2\chi}\right)\,, \qquad (17)$$

$$A = \frac{\alpha\,\mu}{m'\;\sinh\chi}\,.$$

The normalization constant $N_0(\chi)$ in (17) can be obtained (also as in paper [K.A. Milton, I L. Solovtsov, Mod. Phys. Lett. A 16 (2001) 2213] at $\ell = 0$ from the condition

$$\lim_{\alpha\to 0}\varphi_\ell(\rho\,,\chi) = \rho\,p_\ell(\rho\,,\cosh\chi)\xrightarrow[\rho\to\infty]{}\frac{\sin(\rho\chi-\pi\,\ell/2)}{\sinh\chi}\,. \qquad (18)$$

• The generalized power (15) in the solution (16) vanishes at $\rho = i$ for all $\ell \neq 0$. Thus, the expansion for the wave function $\psi_q(\boldsymbol{\rho})$ contains only s-wave ($\ell = 0$). Hence, we can calculate $|\psi_q(i)|^2$, which leads to the following expression for the relativistic S-factor in the case of two particles of unequal masses:

$$S_{\rm uneq}(\chi) = \lim_{\rho \to i} \left| \frac{\varphi_0(\rho\,,\chi)}{\rho} \right| = \frac{X_{\rm uneq}(\chi)}{1 - \exp\left[-X_{\rm uneq}(\chi)\right]}\,,$$
$$X_{\rm uneq}(\chi) = \frac{2\,\pi\,\alpha\,\mu}{m'\,\sinh\chi}\,, \qquad (19)$$

where χ is the rapidity which is related to the total c. m. energy, $\sqrt{s}$, as $(m'^2/\mu)\,\cosh\chi = \sqrt{s}$.

- The function $X_{\rm uneq}(\chi)$ in Eq. (19) can be expressed in terms of the "velocity" u determined by the relation

$$u = \sqrt{1 - \frac{4\,m'^2}{s-(m_1-m_2)^2}}\,, \tag{20}$$

in the form

$$X_{\rm uneq}(u) = \frac{\pi\,\alpha\,\sqrt{1-u^2}}{u}\,. \tag{21}$$

• The L-factor in the nonrelativistic case is defined by derivative of the order ℓ of the the wave function at $r = 0$. In the relativistic case, instead of the derivative, one has to use its finite difference analog [V. G. Kadyshevsky, R. M. Mir-Kasimov, N. B. Skachkov, Nuov. Cim. A. 1968. V. 55. P. 233; Sov. J. Part. Nucl. 1972. V. 2. P. 69]:

$$\Delta^* = \frac{1}{i}\left[\exp\left(i\frac{\partial}{\partial\rho}\right) - 1\right]. \tag{22}$$

Thus, the relativistic L-factor is connected, as one can expect, with the RQP partial wave function $\varphi_\ell(\rho\,,\chi)$ as follows:

$$L_{\,\mathrm{uneq}}(\chi) = \lim_{\rho\to i}\left|\frac{\Gamma(2\,\ell+2)}{(2\,\sinh\,\chi)^\ell\,\Gamma^2(\ell+1)}\left(\Delta^*\right)^\ell\left[\frac{\varphi_\ell(\rho\,,\chi)}{\rho}\right]\right|^2. \tag{23}$$

- $$\varphi_\ell(\rho\,,\chi) = \; N_\ell(\chi)(-\rho)^{(\ell+1)}\, e^{i\rho\chi+iA\chi+i\pi(\ell+1)}\times$$
$$\times\, F\left(\ell+1-iA\,,\ell+1-i\rho\,;2\,\ell+2\,;1-e^{-2\chi}\right)\;. \tag{24}$$

The normalization constant $N_\ell(\chi)$ in Eq. (24) can be obtained (also as in case s-wave, $\ell = 0$) from the condition (18).

By using Eqs. (18), (22), (23) and (24), we finally find the following expression for the relativistic L-factor in the case of two particles of unequal masses:

$$L_{\,\rm uneq}(\chi) = \prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha\,\mu}{m'\;n\;\sinh\chi}\right)^2\right] S_{\,\rm uneq}(\chi)\,. \tag{25}$$

- The relative relativistic velocity of interacting particles, $\mathbf{v}$,

$$|\,\mathbf{v}| = 2\sqrt{\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}}\left(1+\frac{s-(m_1+m_2)^2}{s-(m_1-m_2)^2}\right)^{-1}, \qquad (26)$$

is connected with the "velocity" u by relation

$$|\,\mathbf{v}| = \frac{2\,u}{1+u^2}\,. \qquad (27)$$

From the square of relative 3-momentum $\mathbf{k}'$ of an effective relativistic particle and (27) then follows

$$\mathbf{k}'^2 = (\mu)^2\,(u'_{\rm rel})^2\,, \qquad (28)$$

where

$$u'_{\rm rel} = \frac{2\,u}{\sqrt{1-u^2}} \qquad (29)$$

is the relative velocity of an effective relativistic particle with mass m' and the relative 3-momentum $\mathbf{k}'$ emerging instead of the system of two particles. This result is found to be in full agrement with the physical meaning of Eq. (5), which is a relativistic generalization of the Schrödinger equation in the spirit of Lobachevsky geometry.

- Thus, in terms of relative velocity of an effective relativistic particle (29), the *S*-factor (19) and *L*-factor (25) are given by expressions

$$S_{\,\mathrm{uneq}}(u'_{\mathrm{rel}}) = \frac{X_{\,\mathrm{uneq}}(u'_{\mathrm{rel}})}{1-\exp\left[-X_{\,\mathrm{uneq}}(u'_{\mathrm{rel}})\right]}\,, \qquad (30)$$

$$L_{\,\mathrm{uneq}}(u'_{\mathrm{rel}}) = \prod_{n=1}^{\ell}\left[1+\left(\frac{\alpha}{n\,u'_{\mathrm{rel}}}\right)^{2}\right] S_{\,\mathrm{uneq}}(u'_{\mathrm{rel}})\,, \qquad (31)$$

$$X_{\,\mathrm{uneq}}(u'_{\mathrm{rel}}) = \frac{2\,\pi\,\alpha}{u'_{\mathrm{rel}}}\,.$$

4. Analysis relativistic threshold resummation factor and summing up of the threshold singularities

The S-factor in Eq. (30) only formally has the same form, as the nonrelativistic S-factor (2). However, the S-factor in Eq. (30) has an obviously relativistic nature since as the argument r (the module of radius-vector $\mathbf{r}$) in the Coulomb potential (1) and the relativistic relative velocity of interacting particles, v, [V G. Kadyshevsky, R M. Mir-Kasimov, N B. Skachkov, Sov. J. Part. Nucl. 2 (1972) 69] both are relativistic invariants and hence the relative velocity of an effective relativistic particle (29), according to (28), possesses this property as well.

The relativistic threshold resummation factors (30) and (31) has the following important properties:

• In the nonrelativistic limit, $u \ll 1$, it reproduces the well-known nonrelativistic result.

• In the relativistic limit, $u \to 1$, the factors (30) and (31) go to unity.

• In the case of equal masses factor coincides with S-factor (4) (Milton-Solovtsov).

• The case when one of the particles is at rest means that $m_1 \to \infty$. This give the following of limited of expression for the "velocity":

$$u \xrightarrow[m_1 \to \infty]{} \frac{|\mathbf{k}|}{\sqrt{{m_2}^2 + \mathbf{k}^2} + m_2} \,.$$

• In the ultrarelativistic limit, as it has been argued in papers [W. Lucha, F.F. Schöberl, Phys. Rev. Lett. 64 (1990) 2733; Phys. Lett. B 387 (1996) 573], the bound state spectrum vanishes as mass of an effective relativistic particle $m' \to 0$. This feature reflects an essential difference between potential models and quantum field theory, where an additional dimensional parameter appears Λ. One can conclude that within a potential model, the S- and L-factors which correspond to the continuous spectrum should go to unity in the limit $m' \to 0$.

Thus, in contrast to the nonrelativistic case, the relativistic resummation S- and L-factors (30) and (31), reproduces both the known nonrelativistic and the expected ultrarelativistic limits.

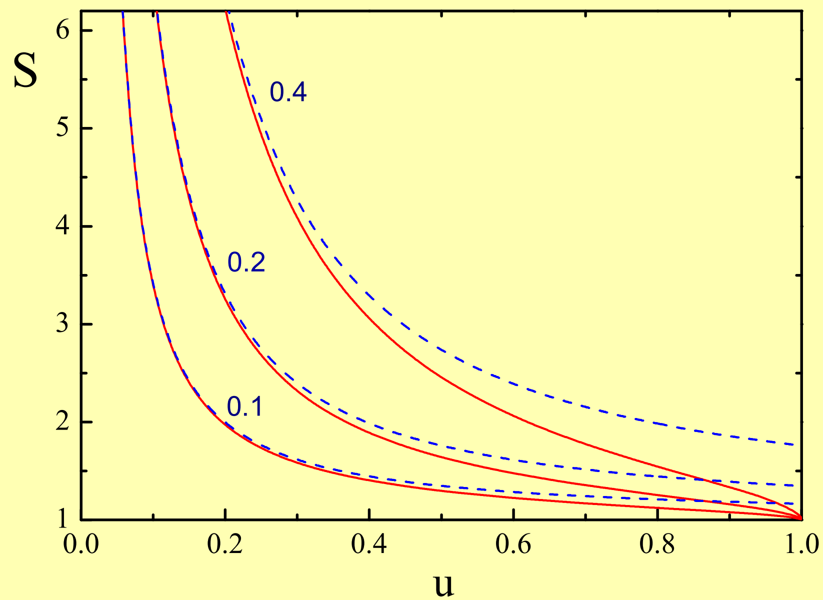
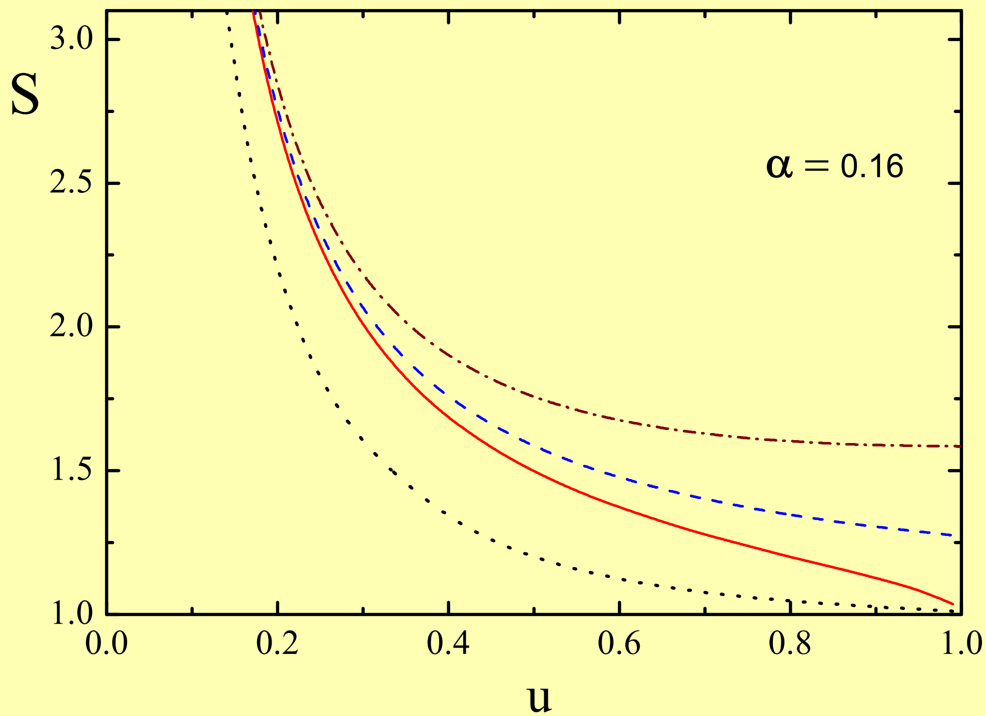
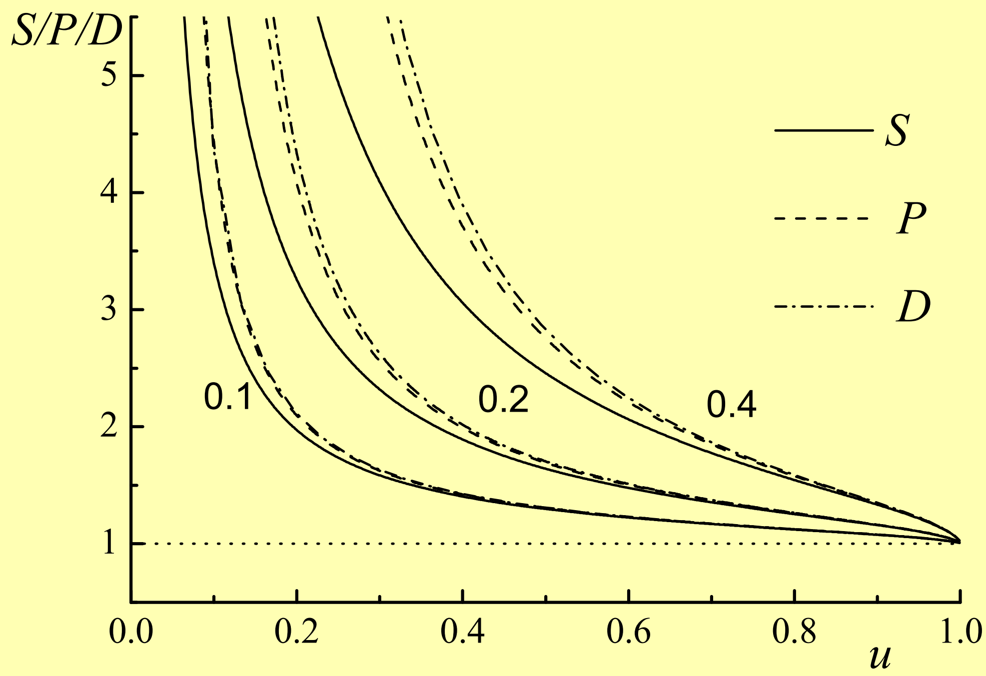


Figure 2 demonstrates the difference in behavior of the nonrelativistic S-factor (2) and the new relativistic S-factor (30) as functions of u at different values of the parameter α (the numbers at the curves). The solid lines correspond to the S-factor in Eq. (30) and the dashed lines to the S-factor (2) with a substitution $v_{\rm nr} \to u$. From this figure one can see that in the region of nonrelativistic values of u, $u \le 0.2$, where the influence of the S-factor is big, the difference between (30) and (2) is practically absent.



S
3.0
2.5
2.0
1.5
1.0
0.0
0.2
0.4
0.6
0.8
1.0
u
α = 0.16



S/P/D
5
4
3
2
1
0.1
0.2
0.4
S
P
D
0.0
0.2
0.4
0.6
0.8
1.0
u

The relativistic factors could have a significant impact in interpreting strong-interaction physics. The $R(s)$-function, determined by the imaginary part of the quark current correlator, occurs as a factor in an integrand, as, for example, for the case of inclusive τ decay, for smearing quantities, and for the Adler D-function. The principal contribution to the function $R(s) \to R_V^{(0)}$ for the vector current with the S-factor can be written as

$$\begin{aligned} R_V^{(0)} = \left[1 - \frac{(m_1 - m_2)^2}{s}\right]^2 \times \\ \times \left[\frac{u\,(3-u^2)}{2} + \frac{(m_1 - m_2)^2}{2s}\,u^3\right] S(u\,,\alpha) \end{aligned} \qquad (32)$$

where the total c. m. energy of interacting particles, $\sqrt{s}$, can be expressed in terms of the "velocity" u as $s = [(m_1+m_2)^2 - (m_1-m_2)^2\,u^2]/(1-u^2)$. The corresponding expression without the S-factor can be found in paper [L.J. Reinders, H R. Rubinstein, and S. Yazaki, Phys. Rep. 1985. V. 127. P. 1].

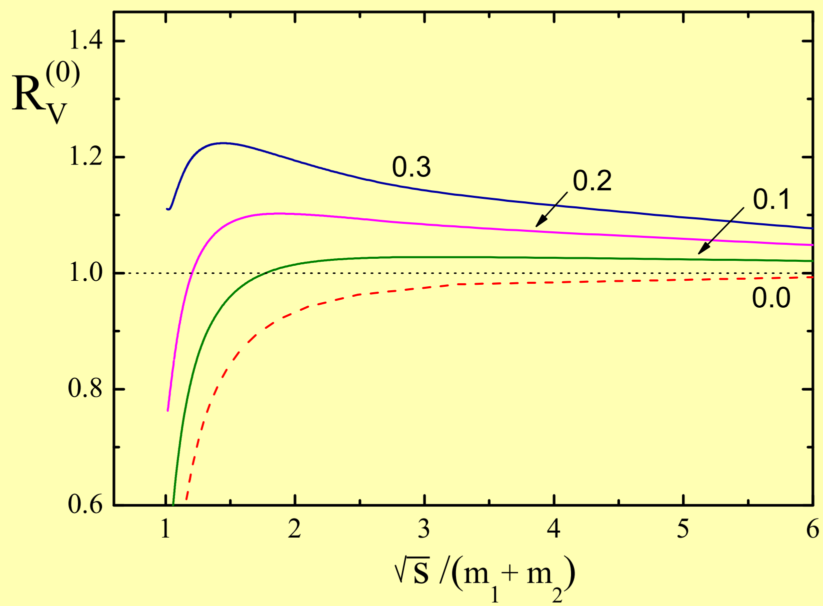
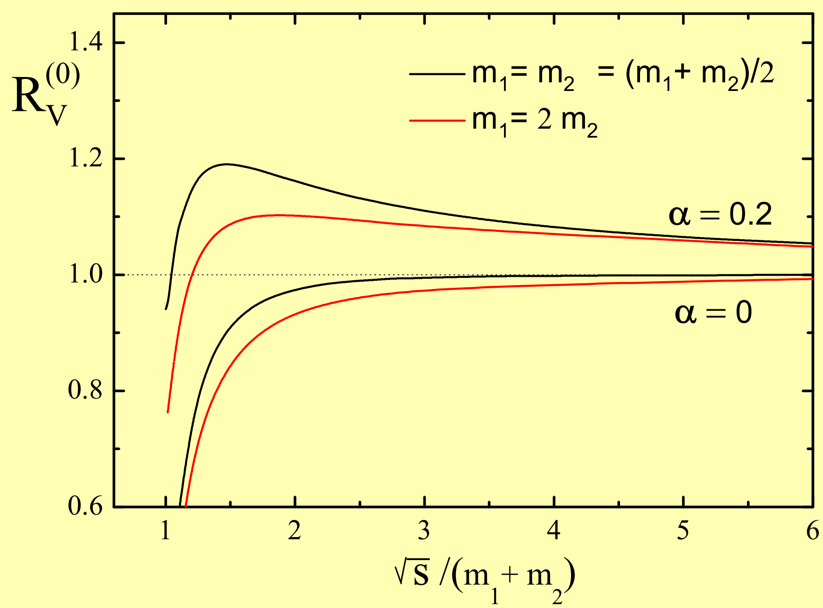


Figure 3 shows the dependence of behavior of value $R_V^{(0)}$ with the new S-factor as a function of dimensionless variable for different values of α (the numbers at the curves). Here a dashed line is the behaviour $R_V^{(0)}$ without S-factor ($\alpha = 0$). We see that the influence of the new S-factor is much stronger in the threshold region and with growing energy $\sqrt{s}$ weakens, and all curves approach unity.



$R_V^{(0)}$
1.4
1.2
1.0
0.8
0.6
1
2
3
4
5
6
$m_1 = m_2 = (m_1 + m_2)/2$
$m_1 = 2\, m_2$
$\alpha = 0.2$
$\alpha = 0$
$\sqrt{s}\,/(m_1 + m_2)$

5. Conclusions

The new relativistic threshold resummation S- and L-factors (30) and (31) for the interaction of two relativistic particles of arbitrary masses were obtained. For this aim the relativistic quasipotential equation in relativistic configuration representation [V.G. Kadyshevsky, M.D. Mateev, R.M. Mir-Kasimov, Yad. Fiz. 11 (1970) 692] with the Coulomb potential for the interaction of two relativistic particles of arbitrary masses was used. The Coulomb potential only formally has the same form as the nonrelativistic potential but differs in the relativistic configuration representation since its behavior corresponds to the quark-antiquark potential $V_{q\bar{q}} \sim \bar{\alpha}_s(Q^2)/Q^2$ with the invariant charge $\bar{\alpha}_s(Q^2) \sim 1/\ln Q^2$. So, the principal effect coming from the running of the QCD coupling is accumulated.

The new S/L-factor coincides in form with the nonrelativistic factor; however, the role of the parameter of velocity is played not by the relative velocity of interacting particles, $\mathbf{v}$, but by the relative velocity (29) of an effective relativistic particle emerging instead of the system of two particles.

The new S/L-factor reproduces both the known nonrelativistic and expected ultrarelativistic limits and correspond to the QCD-like Coulomb potential.

As the new relativistic resummation factors (30) and (31) were obtained within the framework of completely covariant method, one can expect that these factors takes into account more adequately relativistic nature of interaction.

Thank You for Your attention !