Shock wave in the Friedmann–Robertson-Walker space-time

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based on work by I. Ya. Aref'eva, E.O. Pozdeeva and A.A. Bagrov

• According to 't Hooft [1] shock waves in the Minkowski space-time can be used to describe ultra relativistic particles collisions.

• Shock waves in AdS [2] and in dS [3] can be used to describe ultra relativistic particles collisions too.

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[1] G. 't Hooft, *Phys. Lett. B.* **198**, 61, 1987.

[2] S. S. Gubser, S. S. Pufu, A. Yarom, *Phys.Rev.D*, **78**, 2008, 066014

[3] P.C. Aichelburg and R.U. Sexl, *Gen. Relat. and Grav.*, V.2**4**, 1971, 303.

I.Ya. Aref'eva, A.A. Bagrov and E.A. Guseva, *JHEP*, **0912**,009, 2009.

1) P.C. Aicelburg and R.U. Sexl, *Geral Relativity and Gravitation*, V.2**4**, 1971, 303.
2)G. 't Hooft, *Phys. Lett. B.* **198**, 61, 1987.
M. Hotta, M. Tanaka, *Clas. Quan. Grav.*, **10** (1993) 307–314.
3) J. Podolsky, M. Ortaggia, *Clas. Quan. Grav.*, **18** (2001) 2689–2706.
4) G. Esposito, R. Pettorino and P. Scudellaro, *Int. J. Geom. Meth. Mod. Phys.*, **4** (2007), 361.
5) I.Ya. Aref'eva, K.S. Viswanwthan, I.V. Volovich, Nucl. Phys. B **452** (1995) 346–368.
6) J. Choquet–Bruhat, A. Fisher and A. Marsden, Proc. Enrico Fermi Summer School of the Italian Physical Society, Varenna, ed. Y. Ehlers (1978).
7) I.Ya. Aref'eva, A.A. Bagrov and E.A. Guseva, *JHEP*, **0912**,009, 2009.

• In this talk the generalization of this construction for the ultra-relativistic particles in the Friedmann-Robertson-Walker space-time is presented.

• The shock gravitational waves are ultrarelativistic limits to the solutions of classical gravitation theory [4]

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[4] P.C. Aicelburg and R.U. Sexl, *Geral Relativity and Gravitation*, V.2**4**, 1971, 303.

G. Esposito, R. Pettorio and P. Scudellaro, *Int.J.Geom.Meth.Mod.Phys.* **4**,361,2007, arXiv:0606126[gr-qc]

I.Ya. Aref'eva, A.A. Bagrov and L.V. Joukovskaya, *Algebra and analysis* **22(3)**, 3, 2010.

• McVittie metric [5] in cosmological coordinates is

$$dS^2 = -\frac{\left(1-\frac{m}{2a(t)\rho}\right)^2}{\left(1+\frac{m}{2a(t)\rho}\right)^2}dt^2 + a(t)^2\left(1+\frac{m}{2a(t)\rho}\right)^4(\rho^2 d\Omega^2 + d\rho^2),$$

$$d\Omega^2 = \sin^2\theta d\phi^2 + d\theta^2,$$

where $a(t)$ is arbitrary function of t.

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[5] G. C. McVittie, Mon. Not. R. Astron. Soc. 93, 325 (1933).

N. Kalopery, M. Klebanz and D. Martiny, McVittie's Legacy: Black Holes in an Expanding Universe, arXiv:1003.4777[hep-th].

Some interesting cases of function $a(t)$ corresponds to the following types of universes expansion:

- for $a(t) = 1$, the Hubble constant $H = 0$, reduces McVittie metric to the Schwarzschild black hole of mass m,
- for $a(t) = e^{Ht}$, the Hubble constant $H = const$, reduces McVittie metric to de Sitter-Schwarzschild black hole of mass m,
- for $a(t) = k_2 t^n$, the Hubble constant $H = \dfrac{\dot{a}}{a} = \dfrac{n}{t}$.

Shock wave in Minkowski space-time

The Schwarzschild black hole metric in Minkowski space-time:

$$ds_4^2 = -\frac{(1-A^2)}{1+A^2}dt^2 + (1+A)^4(dx^2+dy^2+dz^2), \qquad (1)$$

$$A = \frac{m}{2r}, \quad r^2 = x^2+y^2+z^2.$$

The first order small mass approximation

$$ds_1^2 = ds_{4M}^2 + 4A(ds_{4M}^2 + 2dt^2), \quad ds_{4M} = ds_4|_{A=0}.$$

The Lorenz transformation is

$$t = \gamma(\bar{t} - v\bar{x}), \quad x = \gamma(\bar{t} - v\bar{x}), \quad \gamma = \frac{1}{\sqrt{1-v^2}}.$$

In terms of $\bar{t}$, $\bar{x}$ the function A is

$$A = \frac{p(1-v^2)}{2\sqrt{(\bar{x}-v\bar{t})^2 + (1-v^2)(\bar{y}^2+\bar{z}^2)}}, \text{ where } \quad p = m\gamma$$

and

$$dt^2 = \frac{(d\bar{t} - v d\bar{x})^2}{1-v^2}.$$

Shock wave in Minkowski space-time

$$ds^2_\gamma = ds^2_{4M} + 4p\left(\frac{1}{|\bar{t}-\bar{x}|} - 2\ln(\bar{y}^2+\bar{z}^2)^{1/2}\delta(\bar{t}^2 - \bar{x}^2)\right)(d(\bar{t}-\bar{x}))^2,$$

is obtained by the ultra relativistic limit $\gamma \to \infty$.

Shock wave in dS space-time

The Schwarzschild black hole metric in dS space-time:

$$dS^2 = -\left(1 - \frac{2m}{R} - \frac{R^2}{b^2}\right) dt^2 + \frac{dR^2}{\left(1 - \frac{2m}{R} - \frac{R^2}{b^2}\right)} + $$
$$+ \; R^2(d\theta^2 + \sin^2\theta d\phi^2).$$

The first order small mass approximation of Schwarzschild black hole metric in dS

$$ds^2 = ds^2_{dS} + \frac{2m}{R} dt^2 + \frac{2m}{R} \frac{dR^2}{\left(1 - \frac{R^2}{b^2}\right)^2}, \quad ds^2_{dS} = dS^2|_{m=0}$$

- In the plane coordinates representation the metric is:

$$ds^2 = ds^2_{5M} + ds^2_p, \text{ where } ds^2_{5M} = -dZ_0^2 + \sum_{i=1}^{4} dZ_i^2,$$

$$ds^2_p = \frac{2mb^2}{(Z_4^2 - Z_0^2)^2(b^2 + Z_0^2 - Z_4^2)^{3/2}} \times$$
$$((b^2(Z_4^2 + Z_0^2) + Z_0^2 Z_4^2 - Z_4^4)dZ_0^2 -$$
$$-2(2b^2 + Z_0^2 - Z_4^2)dZ_0 dZ_4 + (b^2(Z_4^2 + Z_0^2) + Z_0^4 - Z_0^2 Z_4^2)dZ_4^2).$$

- The 4D hyperboloid condition to the coordinates in dS:

$$-Z_0^2 + \sum_{i=1}^{4} Z_i^2 = b^2.$$

- The Lorenz transformation along Z_1 coordinate:

$$Z_0 = \gamma(Y_0 + vY_1), \quad Z_1 = \gamma(vY_0 + Y_1). \tag{2}$$

is applied to first order small mass approximation of Schwarzschild black hole in dS with mass rescaling $m = p/\gamma$.

- Shock wave in Minkowski space-time is

$$\begin{aligned} ds^2_\gamma \;=\; & -dY_0^2 + \sum_{i=1}^{4} dY_i^2 + \\ & + \; 4p\left(-2 + \frac{Y_4}{b}\ln\left(\frac{b+Y_4}{b-Y_4}\right)\right)\delta(Y_0+Y_1)(d(Y_0+Y_4))^2. \end{aligned}$$

Shock wave in Friedmann-Robertson-Walker space-time

Coordinates relations

• For description ultrarelativistic particles movement by boost in plane coordinates representation we need in relation 5D Minkowski space-time coordinates with 4D FRW coordinates.

• Connection between four-dimensional spatially flat cosmology and five-dimensional Minkowski space-time has been proposed by M.N. Smolyakov at 2008.

- Consider the 5D Minkowski metric and 4D FRW metric:

$$dS^2_{5M} = -dZ_0^2 + dZ_1^2 + dZ_2^2 + dZ_3^2 + dZ_4^2,\ M_5,\ \text{D=5},$$

$$ds^2_{FRW} = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2),\ \text{FRW, D=4}.$$

- If $a(t)$ is arbitrary function of t, then the hyperboloid condition becomes non-stationary:

$$-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = b^2(t)$$

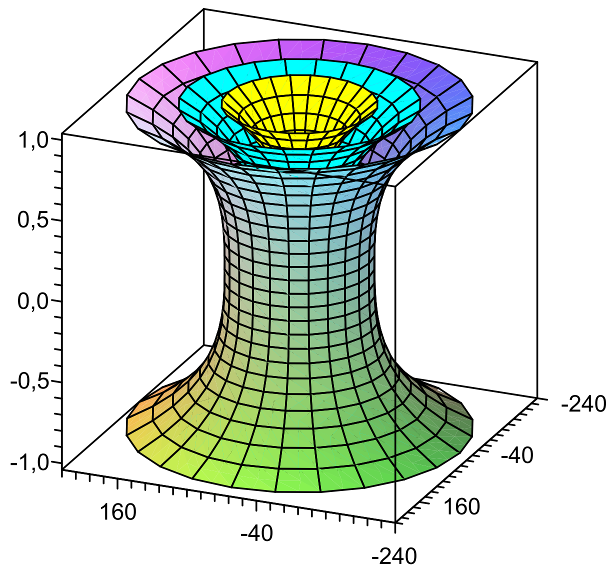

Figure 1: Hyperboloid for different t.

- The surface is defined by:

$$Z_0 = \frac{1}{2}\kappa_1 a(t) - \frac{1}{2}\frac{b^2(t)}{\kappa_1 a(t)} + \frac{1}{2}\frac{a(t)(x^2+y^2+z^2)}{\kappa_1},$$
$$Z_4 = \frac{1}{2}\kappa_1 a(t) + \frac{1}{2}\frac{b^2(t)}{\kappa_1 a(t)} - \frac{1}{2}\frac{a(t)(x^2+y^2+z^2)}{\kappa_1},$$
$$Z_1 = a(t)x, \quad Z_2 = a(t)y, \quad Z_3 = a(t)z.$$

- The metric in 5D Minkowski space-time is equal to metric in 4D FRW, if the following condition relates $a(t)$ with $b(t)$):

$$-\left(\frac{da(t)}{dt}\frac{b(t)}{a(t)}\right)^2 + 2\frac{da(t)}{dt}\frac{db(t)}{dt}\frac{b(t)}{a(t)} + 1 = 0.$$

- In the case $a(t) = \kappa_2 t^n$, we get $b(t) = \pm\frac{t}{\sqrt{n(n-2)}}$.

McVittie metric in small mass approximation

- McVittie metric

$$ds^2 = -\frac{(1-\mu)^2}{(1+\mu)^2}dt^2 + a^2(t)\,(1+\mu)^4\,(dx^2+dy^2+dz^2),$$
$$\mu = \frac{m}{2a(t)\rho}.$$

- First order approximation $(m^2 \sim 0)$,

$$\frac{(1-\mu)^2}{(1+\mu)^2} \approx 1-4\mu, \quad (1+\mu)^4 \approx 1+4\mu,$$

to McVittie's metric is

$$ds_1^2 = ds_{FRW}^2 + 4\mu(ds_{FRW}^2 + 2dt^2).$$

- For $a(t) = k_2 t^n$ the metric can be written in plane coordinates:

$$ds^2 = ds^2_{5M} + \frac{2m}{\sqrt{Z_i^2}}\left(ds^2_{5M} + \frac{2d(Z_0+Z_4)^2}{n^2\kappa_1^2\kappa_2^2(n(n-2)b^2(t))^{n-1}}\right),$$

where $b^2(t) = -Z_0^2 + Z_i^2 + Z_4^2$, $i = \overline{1,3}$.

Lorentz transformation

• Boost in the 5-dimensional Minkowski space-time:

$$Z_0 = \gamma(\widetilde{Z}_0 + v\widetilde{Z}_1), \quad Z_1 = \gamma(\widetilde{Z}_1 + v\widetilde{Z}_0), \quad \gamma = \frac{1}{\sqrt{1-v^2}}.$$

• We apply the Lorentz transformation to the McVittie metric in the first order small mass approximation:

$$ds^2_\gamma = ds^2_{5M} + \frac{2\tilde{m}\left(ds^2_{5M} + 2\frac{d(\gamma(\tilde{Z}_0+v\tilde{Z}_1)+\tilde{Z}_4)^2}{p^2\kappa_1^2\kappa_2^2(p(p-2)b^2(t))^{p-1}}\right)}{\gamma\sqrt{\gamma^2(v\tilde{Z}_0+\tilde{Z}_1)^2+\tilde{Z}_2^2+\tilde{Z}_3^2}}, \ \tilde{m} = m\gamma$$

Lorentz transformation

or

$$ds^2_\gamma = ds^2_{5M} + \frac{2\tilde{m}\left(ds^2_{5M} + 2\frac{d(\gamma(\tilde{Z}_0+v\tilde{Z}_1)+\tilde{Z}_4)^2}{p^2\kappa_1^2\kappa_2^2 t^{2(p-1)}}\right)}{\gamma\sqrt{\gamma^2(v\tilde{Z}_0+\tilde{Z}_1)^2+\tilde{Z}_2^2+\tilde{Z}_3^2}}.$$

- For $\gamma \to \infty$, it is evidently that:

$$ds^2 \mid_{v\to 1} \to ds^2_{5M} + \frac{4\tilde{m}\gamma}{\sqrt{\gamma^2(\tilde{Z}_0+\tilde{Z}_1)^2+\tilde{Z}_2^2+\tilde{Z}_3^2}}\left(\frac{d(\tilde{Z}_0+\tilde{Z}_1)^2}{p^2\kappa_1^2\kappa_2^2 t^{2(p-1)}}\right)$$

Limiting process $\gamma \to \infty$

- Limiting process $\gamma \to \infty$ in generalized function meaning:

$$\int_{-\infty}^{\infty} \frac{\gamma}{\sqrt{\gamma^2 U^2 + X^2}} f(U) dU = f(0) \ln \frac{4\gamma^2}{X^2} + \int_{-\infty}^{\infty} \left(\frac{1}{|U|}\right)_{reg} f(U)\, dU$$

where

$$\int_{-\infty}^{\infty} \left(\frac{1}{|U|}\right)_{reg} f(U)\, dU \equiv$$

$$\equiv \int_{-1}^{1} \frac{f(U) - f(0)}{|U|}\, dU + \int_{-\infty}^{-1} \frac{1}{|U|} f(U) dU + \int_{1}^{\infty} \frac{1}{|U|} f(U) dU.$$

Limiting process $\gamma \to \infty$

The result can be presented by the Dirac-delta function

$$\lim_{\gamma \to \infty} \left[\frac{\gamma}{\sqrt{\gamma^2 U^2 + X^2}} - \delta(U) \ln \gamma^2 \right] = -\delta(U) \ln \frac{X^2}{4} + \left(\frac{1}{|U|} \right)_{reg} .$$

Lorentz transformations in the ultrarelativistic limit the McVittie metric

- After the regularization we have the gravitational waves metric

$$ds^2_{\gamma} = ds^2_{5M} + \frac{4\bar{m}}{p^2\kappa_1^2\kappa_2^2(t)^{2(p-1)}}\delta(U)d(U)^2,\ \bar{m} = \tilde{m}\ln\gamma^2,\ U = Z_0 + Z_1,$$

where

$$t = \left(\frac{Z_0 + Z_4}{k_1 k_2}\right)^{1/n}, \quad t^2 = n(n-2)(-Z_0^2 + Z_1^2 + Z_2^3 + Z_3^2 + Z_4^2)$$

• The obtained metric can be presented with cosmological coordinates:

$$ds^2_\gamma = ds^2_{FRW} + \frac{4\bar{m}}{n^2\kappa_1^2\kappa_2^2(t)^{2(n-1)}}\delta(U)d(U)^2,$$

$$U = \frac{1}{2}k_1k_2t^n - \frac{1}{2}\frac{t^2}{n(n-2)k_1k_2t^n} + \frac{1}{2}\frac{k_2t^n(x^2+y^2+z^2)}{k_1} + k_2t^n x$$

The most interesting case $U = 0$. Shock wave profile $F(U = 0)$ is proportional to $\dfrac{1}{(V/2+Z_4)^{2/n}}$:

$$F(U)|_{u\sim 0} \sim \frac{1}{(V/2+Z_4)^{2/n}}.$$

Conclusion

- It is proposed to use the boosted McVittie metric such as model of ultrarelativistic particle in the Friedmann-Robertson-Walker space-time with $a(t) = kt^n$.

- The shock wave corresponding ultrarelativistic particle in the Friedmann-Robertson-Walker space-time is constructed.

Thank you for attention!