Complex-mass definition and the hypothesis of continuous mass

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The structure of propagators is considered in the framework of the convolution representation. Spectral function was found for a special case when the propagator of scalar unstable particle has Breight-Wigner form. The expressions for the dressed propagators of unstable vector and spinor fields are derived in an analytical way for this case.

1 Introduction

Standard definitions of the mass and width of unstable particles (UP) are closely connected with the structure of the dressed propagator. Traditional way to construct the dressed propagator of UP is Dyson summation. This procedure runs into some problems which are widely discussed in the literature. One of such problems follows from the d'Alembert convergence criterion $|z| < 1$ of the ceries $1/(1-z) = 1 + z + z^2 + ...$, where $z = \Pi_{(1)}(q)/(q^2 - M_0^2)$ and $\Pi_{(1)}(q)$ is the one-particle–irreducible self-energy. The variable z should be correctly redefined before summation, that is we have to perform the renormalization of the $\Pi_{(1)}(q)$ at Lagrangian level. This procedure must be consistent with the infinite Dyson summation and we can not use it after the redefinition at $|z| > 1$. There are, also, the difficulties in the scheme of sequential fixed-order calculations which exhibit themselves in the violation of the gauge invariance. Moreover, using different decompositions of self-energy tensor in the Dyson summation leads to different expressions for vector dressed propagator. Then, the renormalization procedure is connected with the truncation of a Laurent series expansion at the resonance range. So, the renormalized propagator is an approximation of the full one which corresponds to exact two-point function.

The peculiarities of Dyson summation lead to the lack of uniqueness in constructing the propagators of unstable particles. There are several different expressions for the numerator of vector boson propagator $g_{\mu\nu} - q_\mu q_\nu / f(q, M, \Gamma)$, which are exploited in practical calculations. It is known that the commonly used Breit-Wigner (BW) expressions for bosonic and fermionic propagators do not sutisfy the electromagnetic Ward identity [1]. It was shown in Ref. [2,3], that the modified BW propagators

$$D^V_{\mu\nu}(q^2) = \frac{-g_{\mu\nu} + q_\mu q_\nu/(M_V^2 - iM_V\Gamma_V)}{q^2 - (M_V^2 - iM_V\Gamma_V)}; \;\; D_F(\hat{q}) = \frac{\hat{q} + M_F - i\Gamma_F/2}{q^2 - (M_F - i\Gamma_F/2)^2}. \tag{1}$$

sutisfy the electromagnetic Ward identity which provides the gauge invariant description of the processes with UP participation. It was also noted in Ref. [2], that in this case we have to make such modification not

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only in the $q_\mu q_\nu$ term of the propagator, but in the vertexes too. Thus, we get the so-called complex-mass definition which was developed in the frame-work of the complex-mass scheme (CMS) [4–6].

An alternative approach is based on the spectral representation of the propagator of UP [7–14]. It treats UP as a non-perturbative state or effective field (asymptotic free field [13,14]). For the first time, the hypothesis of continuous (smeared) mass of UP was suggested by Matthews and Salam [10]. In the Refs. [13,14], UP is described by the so-called asymptotic free field as the state with indefinite (not fixed) mass. The hypothesis of continuous (smeared, indefinite) mass of UP was developed in a series of works, where quantum field model of UP was represented (see, for instance, review articles [15,16] and references therein).

In this work, we consider the structure of the propagators in the framework of the spectral-representation approach and with account of the Dyson procedure. The paper is organized as follows. In the second section we present the principal elements of the approach under consideration and analyze the general structure of the scalar propagator. The expressions for vector and fermionic propagators are derived in the third and fourth sections respectively. Some conclusions concerning the physical status of the results are made in the last section.

2 Propagator of scalar unstable particle

Propagator of scalar UP can be represented in the following convolution form:

$$D(q) = i\int_{s_0}^{\infty} \frac{\rho(m^2)\,dm^2}{q^2 - m^2 + i\epsilon} = i\int_{s_0}^{\infty} D_0(q^2, m^2)\rho(m^2)dm^2, \tag{2}$$

where $\rho(m^2)$ is spectral function of the parameter m^2, $D_0(q^2, m^2)$ is "bare" scalar propagator and the limit of integration s_0 will be determined further. In the framework of the asymptotic free field approaches (indefinite mass) [13, 14] or the model with continuous mass [15, 16] the expression (2) can be derived directly. In these cases, the field function of scalar UP can be represented in the following convolution form:

$$\phi(x) = \frac{1}{(2\pi)^{3/2}}\int\int \phi(\mathbf{p}, m^2)e^{ipx}d\mathbf{p}\,\omega(m^2)\,dm^2, \tag{3}$$

where $p = (\mathbf{p}, p^0)$, $\phi(\mathbf{p}, m^2)$ is defined in standard way at fixed mass $p^2 = m^2$ and $\omega(m^2)$ is model weight function. Note, the value m is not a conventional observed mass of UP. It is continuous mass parameter which cuts out three-dimentional surface in the four momentum space according to equality $p^2 = m^2$. The canonical commutation relations contain an additional delta-function $\delta(m^2 - m'^2)$. Starting from the standard definition of the Green's function $D(q) = i\int dx\, exp\,(-iqx)\,\langle 0|\hat{T}\phi(x)\phi(0)|0\rangle$, where $\phi(x)$ is defined by (3), by straightforward calculations we get convolution representation of the model propagator (2), where $\rho(m^2) = |\omega(m^2)|^2$. The principal problem of the approach under consideration is to define spectral function $\rho(m^2)$. This definition can be got with the help of the known integration rule

$$\int_a^b \frac{f(x)\,dx}{x \pm i\epsilon} = \mp i\pi f(0) + \mathcal{P}\int_a^b \frac{f(x)}{x}dx, \tag{4}$$

which follows from the Sokhotski-Plemelj formula when $x = 0 \in (a, b)$. In Eq. (4) $\mathcal{P}\int$ stand for the principal part of the integral.

Here, we consider in detail the special case of the spectral function for bosonic UP in the assumption that the scalar propagator has a traditional BW form:

$$D^{BW}(q) = \frac{1}{q^2 - M^2 + iM\Gamma}. \tag{5}$$

To define $\rho(m^2)$ we rewrite Eq.(2) with the help of the integration rule (4). Then, Eq.(2) takes the form:

$$D(q) = -i\pi\rho(q) + \mathcal{P}\int \frac{\rho(m^2)}{q^2 - m^2}\, dm^2 \tag{6}$$

The condition $D(q) = D^{BW}(q)$ leads to a following equalities:

$$\begin{aligned} \Im D(q) &= \, -\pi\rho(q^2) = \frac{-M\Gamma}{[q^2 - M^2]^2 + M^2\Gamma^2}; \\ \Re D(q) &= \mathcal{P}\int \frac{\rho(m^2)\, dm^2}{q^2 - m^2} = \frac{q^2 - M^2}{[q^2 - M^2]^2 + M^2\Gamma^2}, \end{aligned} \tag{7}$$

where the first equalities follow from (6) and the second ones from (5). From the upper equality in (7) it follows

$$\rho(m^2) = \frac{1}{\pi}\frac{M\Gamma}{[m^2 - M^2]^2 + M^2\Gamma^2}. \tag{8}$$

Thus, the condition $\Im D(q) = \Im D^{BW}(q)$ uniquely defines the form of the function $\rho(m^2)$ for the case under consideration (q-independent M and Γ). In Ref. [12] the definition of the function $\rho(m^2)$ was fulfilled in close analogy with above consideration and was finished at this stage. Here, we take into consideration the lower equality of (7) which gives an additional information about the limits of integration. By straightforward calculation we check that the lower equality of Eq.(7) and normalization of the function (8) are realized exactly when $(-\infty < m^2 < \infty)$. So, the parameter m^2 can take a negative value and we have to consider an analytic continuation of the traditional spectral approach.

Now, let us consider the theoretical status of the result and possible consequences of the presence of negative mass parameter $m^2 < 0$ in the integral representations (2) and (3). The form of the spectral function is strictly defined, that is, completely motivated by the choice of the dressed scalar propagator as input condition. It should be noted that appearance of the negative component can be caused by the choice of the BW approximation. However, we do not know correct (exact) expression for the input propagator and evaluate the error of approximation. In the framework of the approaches with indefinite (continuous, smeared) mass the negative component $m^2 < 0$ leads to the states with imaginary mass parameter which usually are interpreted as tachyon states. The problem of the existence of tachyons is under considerable discussion in the last decades. The main attention is paid to the principal problems, such as a violation of causality, tachyon vacuum, and radiation instability. It should be noted, that these problems relate to UP as an observable object with fixed imaginary mass. In the framework of the effective model [16] UP is described by the positive mass square M^2 and we have no the tachyons in the set of physical states. Now we evaluate the contribution of the negative component. The spectral function $\rho(m^2)$ is normalized and can be interpreted as the probability density of parameter m^2. So, the probability of the negative component is:

$$P(m^2 < 0) = \int_{-\infty}^{0} \rho(m^2; M, \Gamma)\, dm^2 \approx \frac{\Gamma}{\pi M}, \;\; (\frac{\Gamma}{M} << 1) \tag{9}$$

From (9) it follows that this probability is proportional to the factor Γ/M which defines the finite-width effects in the processes with UP's participation. This fact can lead to an interesting possible conclusions:

tachyon instability is intrinsic property of UP; it can be interpreted as the cause of unstable particle decay. Now, we evaluate the relative contribution of the negative component to the full propagator which we define as the relation:

$$\epsilon(q^2) = \frac{\int_{-\infty}^{0} D_0(q^2, m^2)\,\rho(m^2)\,dm^2}{\int_{-\infty}^{+\infty} D_0(q^2, m^2)\,\rho(m^2)\,dm^2}. \tag{10}$$

In the expression (10) denominator is full BW propagator (5) and the integration in numerator can be performed directly at $q^2 > 0$. As a result, we get:

$$\epsilon(q^2; M, \Gamma) = \frac{1}{\pi}\,\frac{\Gamma M}{q^2 - M^2 - i\Gamma M}\,[\frac{1}{2}\ln\frac{q^4}{M^2(M^2+\Gamma^2)} + \pi\frac{q^2 - M^2}{\Gamma M}], \tag{11}$$

where we used the approximation $\arctan(M/\Gamma) \approx \pi/2$ in the second term. From (11) it follows strong q^2 -dependence of the relative contribution $\epsilon(q^2; M, \Gamma)$. In particular, at the peak range $\epsilon(M^2; \Gamma, M) \approx -i\Gamma^2/2\pi M^2$, at $q^2 >> M^2$ it has asymptotic $\epsilon(q^2) \to 1$ and at $q^2 << M^2$ from (11) it follows:

$$\epsilon(q^2; M, \Gamma) = \frac{\Gamma}{\pi M}\,[\frac{1}{2}\ln\frac{M^2(M^2+\Gamma^2)}{q^4} + \pi\frac{M}{\Gamma}], \tag{12}$$

So, at small q^2 the value $\epsilon(q^2; M, \Gamma)$ is large and we can not cut off the negative component. At $q^2 < 0$, an upper integral in (10) can be calculated with the help of the integration rule (4) and calculation gives the same effect.

3 Propagator of vector unstable particles

To define the structure of vector propagator, we assume that the spectral function $\rho(m^2)$ is the same as for a scalar UP. Using the standard vector propagator for a free vector particle with a fixed mass, we get:

$$D_{\mu\nu}(q) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{-g_{\mu\nu} + q_\mu q_\nu/(m^2 - i\epsilon)}{q^2 - m^2 + i\epsilon}\,\frac{M\Gamma\,dm^2}{[m^2 - M^2]^2 + M^2\Gamma^2}. \tag{13}$$

In the term $q_\mu q_\nu/(m^2 - i\epsilon)$ we use the same rule of going around pole as in the denominator $q^2 - (m^2 - i\epsilon)$. The integral in Eq. (13) can be evaluated with the help of the formula (4), however, it is easier to do it using the method of contour integration. The integration along the lower contour C_- gives:

$$\begin{aligned} D_{\mu\nu}(q) &= -\frac{M\Gamma}{\pi}\oint_{C_-}\frac{(g_{\mu\nu} - q_\mu q_\nu/(z - i\epsilon))\,dz}{(z - z_-)(z - z_+)(z - z_0)} \\ &= -2iM\Gamma\frac{g_{\mu\nu} - q_\mu q_\nu/(z_-)}{(z_- - z_+)(z_- - z_0)} = \frac{-g_{\mu\nu} + q_\mu q_\nu/(M^2 - iM\Gamma)}{q^2 - M^2 + iM\Gamma}\,. \end{aligned} \tag{14}$$

One can check that the integration along the upper contour C_+ or with the help of the formula (5) leads to the same result. The expression (14) coincides with the well-known expression for modified BW propagator (1) which satisfies to electromagnetic Ward identity [1].

We should note, that both the scalar and vector propagators of UP can be represented in the form with universal complex mass squared:

$$D(q) = \frac{1}{q^2 - M_P^2}; \qquad D_{\mu\nu}(q) = \frac{-g_{\mu\nu} + q_\mu q_\nu/M_P^2}{q^2 - M_P^2}\,, \tag{15}$$

where the structure $M_P^2 = M^2 - iM\Gamma$ usually is called as complex-mass definition. This definition is the base element of the so-called complex-mass scheme of calculation [4,5]. The dressed propagator of a bosonic UP can be formally obtained from the "free" propagator by the substitution $m^2 - i\epsilon \longrightarrow M^2 - iM\Gamma$. So, the infinitesimal value ϵ, which formally defines the rule of going around pole in bare propagator, is an analog of the infinitesimal width of the intermediate state in the framework of the model approach.

4 Propagator of spinor unstable particles

The propagator of a free fermion can be represented in two equivalent forms:

$$\hat{D}(q) = \frac{1}{\hat{q} - m + i\epsilon} = \frac{\hat{q} + m - i\epsilon}{q^2 - (m - i\epsilon)^2}. \tag{16}$$

According to the above mentioned formal rule for constructing the dressed propagator, we have to make the substitution $m - i\epsilon \to M - i\Gamma/2$. Then, the dressed propagator of the spinor UP takes the form (1). Now, we show that the expression (1) can be derived in a more systematic way with the help of the integral representation:

$$\hat{D}(q) = \int \frac{\hat{q} + m - i\epsilon}{q^2 - (m - i\epsilon)^2}\, \rho(m)\, dm\,, \tag{17}$$

where the integration range is not defined yet. The spectral function $\rho(m)$ for fermions differs from the bosonic one, because of another parametrization $M(q) = M_0 + \Re\Sigma(q)$ and $\Gamma(q) = \Im\Sigma(q)$. The spectral function for the case of the spinor UP is as follows:

$$\rho(m) = \frac{1}{\pi}\, \frac{\Gamma/2}{[m - M]^2 + \Gamma^2/4} = \frac{1}{\pi}\frac{\Gamma/2}{(m - M_-)(m - M_+)}, \tag{18}$$

where $M_\pm = M \pm i\Gamma/2$. The main difference between boson and spinor cases is a presence of the linear term m instead the quadratic one m^2, which is defined at the whole real axis $m^2 \in (-\infty, +\infty)$. Here, we consider a straightforward relation between the bosonic parameter range and spinor one. Thus, we have two intervals, naimly $(+i\infty, i0;\ 0, \infty)$ and $(-i\infty, i0;\ 0, \infty)$ for the value m. In the method of contour integration the signs $\pm$ correspond to integration along the contours $C_\pm$, which enclose the first or fourth quadrants of the complex plane. Then, from Eqs. (17) and (18) it follows:

$$\hat{D}_\pm(q) = \pm\frac{\Gamma}{2\pi}\int\limits_{C_\pm} \frac{(\hat{q} + z)\, dz}{(z^2 - z_0^2)(z - z_-)(z - z_+)}\,, \tag{19}$$

where $z_0^2 = q^2 + i\epsilon$, $z_\pm = M_\pm$ and $C_\pm$ are the above described contours. By simple and straightforward calculations we can see that the correct result follows from the integration along the contour C_-, while the integration along the C_+ leads to non-physical result. This is likely caused by the presence of the branch point z_0^2 in the first quadrant. From Eq. (19) it follows:

$$\begin{aligned} \hat{D}_-(q) &= -\frac{\Gamma}{2\pi}\int\limits_{C_-} \frac{dz}{z - z_-}\, \frac{\hat{q} + z}{(z^2 - z_0^2)(z - z_+)} \\ &= -i\Gamma(q)\frac{\hat{q} + z_-}{(z_-^2 - z_0^2)(z_- - z_+)} = \frac{\hat{q} + M - i\Gamma/2}{q^2 - (M - i\Gamma/2)^2}. \end{aligned} \tag{20}$$

The last expression in (20) coincides with the corresponding expression in (1). The spinor complex mass definition differs from the bosonic one, however, it has similar pole-type complex structure. Then, the pole definition of the mass and width of the spinor UP is $M_P = M_\rho - i\Gamma_\rho/2$ in our consideration.

5 Summary

In this work, we have analyzed a special case of the spectral function which follows from the matching the model and standard scalar BW propagator. This function contains the parameters M, Γ and mass variable m^2 prove to be in the interval $(-\infty, +\infty)$. So, the variable m can be imaginary, however, such states have no explicit physical content. It was shown that contribution of the negative component to the full propagator is significant for the deep virtual states. In the framework of this approach we get vector and spinor propagators with the well-known modified BW structure. This structure provides the gauge invariant description and explicitely leads to the complex-mass definition. The q-dependence of the UP mass and width can be introduced into the function $\rho(m^2; \Gamma(q), M(q))$ without the loss of the generality.

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