Relativistic and Vacuum Polarization Corrections to the Lamb Shift in $(\mu_2^3He)^+$, $(\mu_2^4He)^+$

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The Lamb shift $(2P_{1/2} - 2S_{1/2})$ in muonic helium ions $(\mu_2^3He)^+$, $(\mu_2^4He)^+$ is calculated taking into account corrections of orders α^3, α^4, α^5 and α^6. Special attention is given to relativistic corrections with the account of vacuum polarization effects. The obtained shifts 1259.8583 meV ($(\mu_2^3He)^+$) and 1379.1107 meV ($(\mu_2^4He)^+$) can be useful for a comparison with experimental data of the CREMA collaboration.

Muonic helium ions are bound states of negative muon and the nucleus (helion or α-particle). They are among that simple muonic atoms which attract considerable attention in last years due to the CREMA measurements of fine and hyperfine structure in muonic hydrogen, muonic deuterium and muonic helium ions [1–3]. An analysis of all these experiments will allow to clarify the source of the discrepancies in the description of the electron and muon atoms. For successful realization of the CREMA program and finding more accurate values of charge radii of the proton, deuteron, helion and α-particle it is important to perform a precise calculation of the transition frequencies between the levels 2S and 2P [4–6] The aim of this work is to present the Lamb shift $(2P - 2S)$ calculation in muonic helium ions with the precision 0.001 meV. For a solution of this task we calculate different corrections of orders $\alpha^4 \div \alpha^6$ which are determined by relativistic, vacuum polarization effects, nuclear structure corrections in first, second and third orders of perturbation theory.

Our approach to the calculation of the Lamb shift corrections in muonic helium ions is based on the quasi-potential method in quantum electrodynamics [7–9]. The two-particle bound state is described by the Schrödinger equation. The leading order contribution to the particle interaction operator is determined by the Breit Hamiltonian ($\delta_I = 1 (\delta_I = 0)$ for nuclei with a half-integer (integer) spin) [10]:

$$H_B = \frac{\mathbf{p}^2}{2\mu} - \frac{Z\alpha}{r} - \frac{\mathbf{p}^4}{8m_1^3} - \frac{\mathbf{p}^4}{8m_2^3} + \frac{\pi Z\alpha}{2}\left(\frac{1}{m_1^2} + \frac{\delta_I}{m_2^2}\right)\delta(\mathbf{r}) - \tag{1}$$

$$-\frac{Z\alpha}{2m_1m_2r}\left(\mathbf{p}^2 + \frac{\mathbf{r}(\mathbf{r}\mathbf{p})\mathbf{p}}{r^2}\right) + \frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2} + \frac{1}{2m_1m_2}\right)(\mathbf{L}\sigma_1) = H_0 + \Delta V^B,$$

where $H_0 = \mathbf{p}^2/2\mu - Z\alpha/r$; m_1, m_2 are the masses of the muon and nucleus (helion or α-particle) and $\mu = m_1m_2/(m_1+m_2)$. The wave functions of $2S$ and $2P$ states are the following:

$$\psi_{200}(r) = \frac{W^{3/2}}{2\sqrt{2\pi}} e^{-\frac{Wr}{2}}\left(1 - \frac{Wr}{2}\right),\ \psi_{2lm}(r) = \frac{W^{3/2}}{2\sqrt{6}} e^{-\frac{Wr}{2}} Wr Y_{lm}(\theta,\phi),\ W = \mu Z\alpha. \tag{2}$$

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The effects of vacuum polarization (VP) lead to different corrections to the Breit Hamiltonian 1. The one-loop VP corrections to the Breit interaction are a subclass of interactions that can be called relativistic corrections taking into account the vacuum polarization. They are determined by expressions obtained in [6, 11]:

$$\Delta V^B_{VP}(r) = \frac{\alpha}{3\pi}\int_1^\infty \rho(\xi)d\xi \sum_{i=1}^4 \Delta V^B_{i,VP}(r), \tag{3}$$

$$\Delta V^B_{1,VP} = \frac{Z\alpha}{8}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\left[4\pi\delta(\mathbf{r}) - \frac{4m_e^2\xi^2}{r}e^{-2m_e\xi r}\right], \tag{4}$$

$$\Delta V^B_{2,VP} = -\frac{Z\alpha m_e^2\xi^2}{m_1m_2r}e^{-2m_e\xi r}(1-m_e\xi r), \tag{5}$$

$$\Delta V^B_{3,VP} = -\frac{Z\alpha}{2m_1m_2}p_i\frac{e^{-2m_e\xi r}}{r}\left[\delta_{ij}+\frac{r_ir_j}{r^2}(1+2m_e\xi r)\right]p_j, \tag{6}$$

$$\Delta V^B_{4,VP} = \frac{Z\alpha}{r^3}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)e^{-2m_e\xi r}(1+2m_e\xi r)(\mathbf{L}\sigma_1). \tag{7}$$

The matrix elements (3) over wave functions (2) give corrections in first order perturbation theory:

$$\Delta E^B_{1,VP}(2P-2S) = \begin{cases} -0.8670\ meV \\ -0.8931\ meV \end{cases}, \tag{8}$$

$$\Delta E^B_{2,VP}(2P-2S) = \begin{cases} 0.0150\ meV \\ 0.0116\ meV \end{cases}, \tag{9}$$

$$\Delta E^B_{3,VP}(2P-2S) = \begin{cases} 0.0281\ meV \\ 0.0219\ meV \end{cases}, \tag{10}$$

$$\Delta E^B_{4,VP}(2P-2S) = \begin{cases} -0.0860\ meV \\ -0.0876\ meV \end{cases}. \tag{11}$$

Terms $\Delta V^B_{2,VP}$, $\Delta V^B_{3,VP}$, $\Delta V^B_{4,VP}$ account for recoil effects in the ratio m_1/m_2. Two-loop vacuum polarization corrections of order $\alpha^2(Z\alpha)^4$ appear as a result of two-loop modification of the Breit Hamiltonian.

In second order perturbation theory we have a number of vacuum polarization corrections of orders $\alpha^2(Z\alpha)^2$ and $\alpha(Z\alpha)^4$:

$$\Delta E^{VP}_{SOPT} = <\psi|\Delta V^C_{VP}\tilde{G}\Delta V^C_{VP}|\psi> + 2<\psi|\Delta V^B\tilde{G}\Delta V^C_{VP}|\psi>, \tag{12}$$

where the reduced Coulomb Green's function (RCGF)

$$\tilde{G}_n(\mathbf{r},\mathbf{r}') = \sum_{l,m}\tilde{g}_{nl}(r,r')Y_{lm}(\mathbf{n})Y^*_{lm}(\mathbf{n}'). \tag{13}$$

The radial function $\tilde{g}_{nl}(r,r')$ can be presented in the form of Sturm expansion in Laguerre polynomials. In the calculation of separate matrix elements it is convenient to use reduced Coulomb Green's functions in the form obtained in [12]:

$$\tilde{G}(2S) = -\frac{Z\alpha\mu^2}{4x_1x_2}e^{-\frac{x_1+x_2}{2}}\frac{1}{4\pi}g_{2S}(x_1,x_2), \tag{14}$$

$$g_{2S}(x_1,x_2)=8x_<-4x_<^2+8x_>+12x_<x_>-26x_<^2x_>+2x_<^3x_>-4x_>^2-26x_<x_>^2+23x_<^2x_>^2- \tag{15}$$
$$-x_<^3x_>^2+2x_<x_>^3-x_<^2x_>^3+4e^x(1-x_<)(x_>-2)x_>+4(x_<-2)x_<(x_>-2)x_>\times$$
$$\times[-2C+Ei(x_<)-\ln(x_<)-\ln(x_>)],$$

$$\tilde{G}(2P)=-\frac{Z\alpha\mu^2}{36x_1^2x_2^2}e^{-\frac{x_1+x_2}{2}}\frac{3}{4\pi}\frac{(\mathbf{x}_1\mathbf{x}_2)}{x_1x_2}g_{2P}(x_1,x_2), \tag{16}$$

$$g_{2P}(x_1,x_2)=24x_<^3+36x_<^3x_>+36x_<^3x_>^2+24x_>^3+36x_<x_>^3+36x_<^2x_>^3+49x_<^3x_>^3-3x_<^4x_>^3- \tag{17}$$
$$-12e_<^x(2+x_<+x_<^2)x_>^3-3x_<^3x_>^4+12x_<^3x_>^3[-2C+Ei(x_<)-\ln(x_<)-\ln(x_>)],$$

where $x_<=min(x_1,x_2)$, $x_>=max(x_1,x_2)$, $C=0.57721566...$ is the Euler constant. As a result two-loop VP contribution of first term in (12) can be written in integral form. Subsequent integration over particle coordinates and spectral parameter gives the following result:

$$\Delta E_{SOPT}^{VP,VP}(2S)=-\frac{\mu\alpha^2(Z\alpha)^2}{72\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\times \tag{18}$$

$$\times\int_0^\infty\left(1-\frac{x}{2}\right)e^{-x\left(1-\frac{2m_e\xi}{W}\right)}dx\int_0^\infty\left(1-\frac{x'}{2}\right)e^{-x'\left(1-\frac{2m_e\eta}{W}\right)}dx'g_{2S}(x,x')=\begin{cases}-1.8640\ meV\\-1.9017\ meV'\end{cases}$$

$$\Delta E_{SOPT}^{VP,VP}(2P)=-\frac{\mu\alpha^2(Z\alpha)^2}{7776\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\times \tag{19}$$

$$\times\int_0^\infty e^{-x\left(1+\frac{2m_e\xi}{W}\right)}dx\int_0^\infty e^{-x'\left(1+\frac{2m_e\eta}{W}\right)}dx'g_{2P}(x,x')=\begin{cases}-0.1867\ meV\\-0.1942\ meV'\end{cases}$$

Second term in (12) has a similar structure. A transformation of different matrix elements in it can be carried out by means of following algebraic relations:

$$<\psi|\frac{\mathbf{p}^4}{(2\mu)^2}\sum\nolimits_m'\frac{|\psi_m><\psi_m|}{E_2-E_m}\Delta V_{VP}^C|\psi>=<\psi|(E_2+\frac{Z\alpha}{r})(\hat{H}_0+\frac{Z\alpha}{r})\sum\nolimits_m'\frac{|\psi_m><\psi_m|}{E_2-E_m}\Delta V_{VP}^C|\psi>= \tag{20}$$
$$=<\psi|\left(E_2+\frac{Z\alpha}{r}\right)^2\tilde{G}\Delta V_{VP}^C|\psi>-<\psi|\frac{Z\alpha}{r}\Delta V_{VP}^C|\psi>+<\psi|\frac{Z\alpha}{r}|\psi><\psi|\Delta V_{VP}^C|\psi>.$$

Three terms in right part of (20) can be calculated using expressions for wave functions and perturbation potentials. So, for example, in the case of ${}_2^4He$ their numerical values are equal to (-2.0398) meV, 0.1017 meV, (-1.0576) meV, (-0.0600) meV, 0.0201 meV, (-0.0653) meV for 2S and 2P-states. Integral expressions of corrections from three other terms of the Breit potential (1) are:

$$\Delta E_{SOPT}^{VP,(1)}(2S)=\frac{\alpha(Z\alpha)^4\mu^3}{48\pi}\left(\frac{1}{m_1^2}+\frac{\delta_I}{m_2^2}\right)\int_1^\infty\rho(\xi)d\xi\times \tag{21}$$

$$\int_0^\infty\left(1-\frac{x}{2}\right)e^{-x(1+2m_e\xi/W)}[4x(x-2)(\ln x+C)+x^3-13x^2+6x+4]=\begin{cases}-1.3611\ meV\\-1.4028\ meV'\end{cases}$$

$$\Delta E^{VP,(2)}_{SOPT}(2P) = -\frac{\alpha(Z\alpha)^4\mu^3}{648\pi}\left(\frac{1}{4m_1^2}+\frac{1}{2m_1m_2}\right)\int_1^\infty \rho(\xi)d\xi\int_0^\infty \frac{dx}{x^2}e^{-x}\times \tag{22}$$

$$\int_0^\infty dx' e^{-x'(1+2m_e\xi/W)}g_{2P}(x,x') = \begin{cases} -0.0935\ meV \\ -0.0958\ meV \end{cases},$$

$$\Delta E^{VP,(3)}_{SOPT}(2P) = -\frac{\alpha(Z\alpha)^4\mu^3}{2592\pi m_1m_2}\int_1^\infty \rho(\xi)d\xi\int_0^\infty dx e^{-x(1+2m_e\xi/W)}\times \tag{23}$$

$$\int_0^\infty \frac{dx'}{x'}g_{2P}(x,x')e^{-x'/2}\left(\frac{2}{x'}-\frac{1}{4}-\frac{d^2}{dx'^2}\right)x'e^{-x'/2} = \begin{cases} -0.0084\ meV \\ -0.0066\ meV \end{cases},$$

$$\Delta E^{VP,(3)}_{SOPT}(2S) = -\frac{\alpha(Z\alpha)^4\mu^3}{24\pi m_1m_2}\int_1^\infty \rho(\xi)d\xi\int_0^\infty dx e^{-x(1+2m_e\xi/W)}\left(1-\frac{x}{2}\right)\times \tag{24}$$

$$\int_0^\infty dx' g_{2S}(x,x')e^{-x'/2}\left(\frac{2}{x'}-\frac{1}{4}-\frac{d^2}{dx'^2}\right)e^{-x'/2}\left(1-\frac{x'}{2}\right) = \begin{cases} -0.1064\ meV \\ -0.0829\ meV \end{cases}.$$

Omitting further details of the calculation in (20) we present summary numerical contribution to the shift $(2P-2S)$ from second term in (12) obtained from all terms of the Breit potential (1):

$$\Delta E^{B,VP}_{SOPT}(2P-2S) = \begin{cases} 1.4192\ meV \\ 1.4682\ meV \end{cases}. \tag{25}$$

Three-loop VP contributions in second order PT can be calculated in a similar way. Corresponding potentials are written explicitly in [13]. Accounting an accuracy of the calculation we present them here only for 2S-state:

$$\Delta E^{VP-VP,VP}_{SOPT}(2S) = -\frac{\mu\alpha^3(Z\alpha)^2}{108\pi^3}\int_1^\infty \rho(\xi)d\xi\int_1^\infty \rho(\eta)d\eta\int_1^\infty \rho(\zeta)d\zeta\int_0^\infty dx(1-\frac{x}{2})\times \tag{26}$$

$$\int_0^\infty dx'(1-\frac{x'}{2})e^{-x'(1+\frac{2m_e\zeta}{W})}\frac{1}{\xi^2-\eta^2}\left[\xi^2 e^{-x(1+\frac{2m_e\xi}{W})}-\eta^2 e^{-x(1+\frac{2m_e\eta}{W})}\right]g_{2S}(x,x') = \begin{cases} -0.0104\ meV \\ -0.0107\ meV \end{cases},$$

$$\Delta E^{2-loop\ VP,VP}_{SOPT}(2S) = -\frac{\mu\alpha^3(Z\alpha)^2}{18\pi^3}\int_0^1 \frac{f(v)dv}{1-v^2}\int_1^\infty \rho(\xi)d\xi\times \tag{27}$$

$$\times\int_0^\infty dx\left(1-\frac{x}{2}\right)e^{-x(1+\frac{2m_e}{\sqrt{1-v^2}W})}\int_0^\infty dx'\left(1-\frac{x'}{2}\right)e^{-x'(1+\frac{2m_e\xi}{W})}g_{2S}(x,x') = \begin{cases} -0.0168\ meV \\ -0.0171\ meV \end{cases},$$

In the evaluation of these and other corrections in SOPT we have two characteristic integrals:

$$I_1 = \int_0^\infty g_{2S}(x,y)\left(1-\frac{x}{2}\right)e^{-a_1x}dx = \frac{1}{a_1^5}\Big[2a_1^3(y(4\gamma(y-2)+(y-13)y+6)+4)- \tag{28}$$

$$4a_1^2(y(4\gamma(y-2)+(y-15)y+10)+4)-4(2(a_1-2)a_1+3)a_1(y-2)y(\mathrm{Ei}(-y(a_1-1))-\ln\frac{(a_1-1)y}{a_1})+$$
$$a_1(y(12\gamma(y-2)+y(3y-53)+46)+12)+12(y-2)y+4a_1e^{-y(a_1-1)}(a_1(a_1(y-2)-y+4)+3(y-1))\Big],$$
$$I_2=\int_0^\infty g_{2P}(x,y)e^{-a_1x}xdx=\frac{1}{(a_1-1)a_1^6}\Big[-72\Big(a_1^2(y(y(y(y+4\gamma-9)-12)-12)-8)+ \quad (29)$$
$$4(a_1-1)a_1y^3(-\mathrm{Ei}(y-a_1y)+\ln\frac{(a_1-1)y}{a_1})+a_1(y(12-y(y(y+4\gamma-13)-12))+8)-5y^3\Big)-$$
$$72a_1e^{-y(1-a_1)}\Big(a_1^4y^3+4\Big(a_1^3-1\Big)y^2+4(a_1-1)(2a_1+1)y+8(a_1-1)\Big)\Big].$$

The contribution of three-loop VP in third order PT is determined by the sum of two terms:

$$\Delta E=<\psi_2|\Delta V^C\tilde G\Delta V^C\tilde G\Delta V^C|\psi_2>-<\psi_2|\Delta V^C|\psi_2><\psi_2|\Delta V^C\tilde G\tilde G\Delta V^C|\psi_2>. \quad (30)$$

Let us write explicitly the matrix elements for $2S$ state:

$$\Delta E_{TOPT,1}(2S)=-\frac{\mu Z^2\alpha^5}{864\pi^3}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\int_1^\infty\rho(\zeta)d\zeta\int_0^\infty\Big(1-\frac{x}{2}\Big)e^{-x(1+2m_e\xi/W)}dx\times \quad (31)$$
$$\int_0^\infty\Big(1-\frac{x''}{2}\Big)e^{-x''(1+2m_e\zeta/W)}dx''\int_0^\infty\frac{dx'}{x'}e^{-x''(1+2m_e\zeta/W)}g(x,x')g(x',x'')=\begin{cases}-0.0044\ meV\\-0.0045\ meV\end{cases},$$
$$\Delta E_{TOPT,2}(2S)=\frac{\alpha^2}{288\pi^2}\int_1^\infty\rho(\xi)d\xi\int_1^\infty\rho(\eta)d\eta\int_0^\infty\Big(1-\frac{x}{2}\Big)e^{-x(1+2m_e\xi/W)}dx\times \quad (32)$$
$$\int_0^\infty\Big(1-\frac{x''}{2}\Big)e^{-x''(1+2m_e\eta/W)}dx''\int_0^\infty dx'g(x,x')g(x',x'')\begin{cases}2041.9990\ meV\\2077.2217\ meV\end{cases}=\begin{cases}0.0037\ meV\\0.0038\ meV\end{cases}.$$

In the same way we calculate other two and three-loop VP corrections accounting for recoil effects and nuclear structure corrections. Numerical values of the Lamb shift in muonic helium-3 ion and muonic helium-4 ion are equal to 1259.8583 meV and 1379.1107 meV respectively [13]. These values can be considered as reliable estimates when compared to experimental data of the CREMA collaboration. The reliability of such comparison is provided by the following features of our calculation:

1. Since numerical value of parameter $m_e/\mu Z\alpha=0.34$ for muonic helium ions, the VP effects give the main contribution to the Lamb shift. We consider one-, two-, three-loop VP corrections in first, second and third orders of perturbation theory.

2. We calculate complicated corrections connected with VP and relativistic effects.

3. The nuclear structure effects are determined in terms of the charge radius of the nucleus and with the help of the electromagnetic form factors.

4. We take into account the results of calculations of a large number of authors [5, 14–16].

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