Recent developments in Neutrino Physics

Davide Meloni[1]

Dipartimento di Matematica e Fisica,
Universitá di Roma Tre
Via della Vasca Navale 84, 00144 Rome, Italy

We discuss the recent progress in neutrino physics in the following fields: (i) interpretation of the short and long neutrino oscillation data in terms of neutrino flavor transitions; (ii) models for neutrino masses and mixings.

1 Introduction

The main recent interesting result on neutrino mixing has been the measurement of θ_{13} by T2K [1], MINOS [2], DOUBLE CHOOZ [3], RENO [4] and DAYA-BAY [5] experiments. Global fits of the oscillation parameters [6], [7], [8], summarized in Tab.1 for the normal ordering of the neutrino mass eigenstates only, show that the combined value of $\sin^2\theta_{13}$ is about 10 σ away from zero and that its central value is rather large, close to the previous upper bound. For the other mixing parameters, there are solid indications of the deviation of θ_{23} from the maximal value, probably in the first octant and, thanks to the combined T2K and DAYA-BAY data, a tenuous hints for non-zero δ is starting to appear from the data.

Quantity	Ref. [6]	Ref. [7]	Ref. [8]
Δm^2_{sun} (10^{-5} eV^2)	$7.54^{+0.26}_{-0.22}$	$7.50^{+0.19}_{-0.17}$	$7.60^{+0.19}_{-0.18}$
Δm^2_{atm} (10^{-3} eV^2)	2.43 ± 0.06	2.447 ± 0.0047	$2.48^{+0.05}_{-0.07}$
$\sin^2\theta_{12}$	0.308 ± 0.017	$0.304^{+0.013}_{-0.012}$	0.323 ± 0.016
$\sin^2\theta_{23}$	$0.437^{+0.033}_{-0.023}$	$0.452^{+0.052}_{-0.028}$	$0.567^{+0.032}_{-0.124}$
$\sin^2\theta_{13}$ (10^{-2})	$2.34^{+0.20}_{-0.19}$	2.18 ± 0.10	2.26 ± 0.12
δ/π	$1.39^{+0.38}_{-0.27}$	$0.85^{+0.11}_{-0.19}$	$1.41^{+0.55}_{-0.40}$

Table 1: *Fits to neutrino oscillation data.*

The interpretation of neutrino appearance and disappearance data in terms of 3ν oscillation is quite robust and will be revised in Sect.2. There exists, however, a handful of short baseline data that do not fit into this scheme and demand, among several possibilities, the presence of one or more sterile states. These anomalies and their connections with extended models of neutrino oscillations will be discussed in Sect.3. Although the leptonic CP violating phase δ, the octant of θ_{23} and the mass hierarchy are not completely determined by the data, the results shown in Tab.1 indicate that all other mixing parameters are very well constrained; thus, the necessity of focusing on plausible explanation of these values in theoretical framework beyond the current formulation of particle theory is mandatory. Some attempts in this direction will be reviewed in Sect.4.

[1]meloni@fis.uniroma3.it

2 The standard formulation of neutrino oscillation

The presence of non-zero masses for the light neutrinos call for the introduction of the $n \times n$ leptonic mixing matrix, $U \equiv U_{PMNS}$ [9], which connects the flavor eigenstates with the mass eigenstates:

$$\nu_\alpha = \sum_i U_{\alpha i} \nu_i \,, \tag{1}$$

where α denotes one of the active neutrino flavors ($e, \ \mu$ or τ) while i runs over the light mass eigenstate labels. Being a unitary matrix, U contains, after suitable rotations of the leptonic fields, $1/2 \ n \cdot (n-1)$ physical angles and $1/2 \ (n-1) \cdot (n-2)$ physical phases. The neutrino mass differences (Δm^2_{ij}) and the mixing parameters ($\theta_{ij}, \ \delta$) can be probed by studying oscillations between different flavors of neutrinos, as a function of the neutrino energy E and the traveled distance L. In fact, the oscillation probability $P(\nu_\alpha \to \nu_\beta)$ is given by the absolute square of the overlap of the observed flavor state, $|\nu_\beta\rangle$, with the time-evolved initially-produced flavor state, $|\nu_\alpha\,(t)\rangle$. In vacuum, the evolution operator involves just the Hamiltonian H_0 of a free particle, yielding the well-known result:

$$\begin{aligned} P(\nu_\alpha \to \nu_\beta) = & \ \left|\langle \nu_\beta | e^{-iH_0 L} | \nu_\alpha \rangle\right|^2 = \textstyle\sum_{i,j} U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j} e^{-i\Delta m^2_{ij} L/2E} \\ = & \ P_{\mathrm{CP-even}}(\nu_\alpha \to \nu_\beta) + P_{\mathrm{CP-odd}}(\nu_\alpha \to \nu_\beta)\,, \end{aligned} \tag{2}$$

where we used the standard approximation that $|\nu\rangle$ is a plane wave, $|\nu_i(t)\rangle = e^{-i\,E_i t}|\nu_i(0)\rangle$ and the fact that neutrinos are relativistic:

$$E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2E_i}\,. \tag{3}$$

The CP-even and CP-odd contributions are:

$$\begin{aligned} P_{\mathrm{CP-even}}(\nu_\alpha \to \nu_\beta) = & \ P_{\mathrm{CP-even}}(\overline{\nu}_\alpha \to \overline{\nu}_\beta) \\ = & \ \delta_{\alpha\beta} - 4 \textstyle\sum_{i>j} \ Re\,(U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}) \sin^2\left(\frac{\Delta m^2_{ij} L}{4E}\right) \\ P_{\mathrm{CP-odd}}(\nu_\alpha \to \nu_\beta) = & \ -P_{\mathrm{CP-odd}}(\overline{\nu}_\alpha \to \overline{\nu}_\beta) \\ = & \ 2 \textstyle\sum_{i>j} \ Im\,(U_{\alpha i} U^*_{\beta i} U^*_{\alpha j} U_{\beta j}) \sin\left(\frac{\Delta m^2_{ij} L}{2E}\right) \end{aligned} \tag{4}$$

so that

$$P(\overline{\nu}_\alpha \to \overline{\nu}_\beta) = P(\nu_\beta \to \nu_\alpha) = P_{\mathrm{CP-even}}(\nu_\alpha \to \nu_\beta) - P_{\mathrm{CP-odd}}(\nu_\alpha \to \nu_\beta) \tag{5}$$

where, by CPT invariance, $P(\nu_\alpha \to \nu_\beta) = P(\overline{\nu}_\beta \to \overline{\nu}_\alpha)$. In eq.(4), $\Delta m^2_{ij} = m(\nu_j)^2 - m(\nu_i)^2$, and the combination $\Delta m^2_{ij} L/(4E)$ in $\hbar = c = 1$ units can be replaced by $1.27\,\Delta m^2_{ij} L/E$ with Δm^2_{ij} in eV^2 and $(L, \ E)$ in (Km, GeV).

For $\beta = \alpha$ (disappearance experiments) no CP-violation can appear since the product of the mixing matrix elements is real. On the contrary, for $\beta \neq \alpha$ (appearance experiments), a CP-violating term could be observed. However, at distances L large compared to all the individual oscillation lengths, $\lambda^{\mathrm{osc}}_{ij} \sim E/\Delta m^2_{ij}$, the sine squared terms in $P_{\mathrm{CP-even}}$ average to 0.5 whereas the sine terms in $P_{\mathrm{CP-odd}}$ average to zero. Therefore CP violating effects are largest and hence easiest to observe only at distances in between the smallest (where the sinus term in $P_{\mathrm{CP-odd}}$ can be expanded in a series power of its argument) and the largest oscillation lengths.

Neutrino oscillations in matter may differ from oscillations in vacuum significantly. The effects of the neutrino interactions with matter manifest themselves in the oscillation probability in the Mikheyev - Smirnov - Wolfenstein (MSW) effect [10]: even if the mixing angle θ in vacuum is very small, matter can enhance neutrino mixing and the probabilities of neutrino oscillations if parameters are carefully chosen. This effect can be very important for neutrinos propagating in the Sun and in the Earth as well as for neutrino oscillations in supernovae. Matter effects could be due to the fact that neutrinos can be absorbed by the matter constituents, or scattered off them, changing their momentum and energy. Neutrinos can also experience forward scattering, an elastic scattering in which their momentum is not changed. This process is coherent and creates mean potentials V_a for neutrinos which are proportional to the number of the scatterers.

The effective Hamiltonian that describes the interactions of neutrinos in matter takes into account the neutral current (NC) and the charged current (CC) interactions with protons and neutrons, resulting in the following expressions for the effective potentials [11]:

$$V_e = \sqrt{2}\, G_F \left(N_e - \frac{N_n}{2} \right), \qquad V_\mu = V_\tau = \sqrt{2}\, G_F \left(-\frac{N_n}{2} \right) \tag{6}$$

where N_e and N_n are respectively the electron and neutron number density.

The common term due to NC interactions appears as a diagonal entry in the interaction Hamiltonian (as it is clear from eq.(6)), and therefore it does not contribute to the flavor states evolution. The evolution equation for the system can be written as follows[2]:

$$i\frac{d\nu_\alpha}{dt} = \sum_\beta \left[\left(p + \frac{m_1^2}{2\,E_\nu} \right) \delta_{\alpha\beta} + \left(\sum_{j>1} U_{\alpha j} U^*_{\beta j} \frac{\Delta m^2_{1j}}{2E_\nu} \right) + A\, \delta_{\alpha e} \delta_{\beta e} \right] \nu_\beta\,, \tag{7}$$

where we introduced the constant

$$A = \sqrt{2} G_F N_e = 0.76 \times 10^{-4}\, \mathrm{eV}^2\, Y_e\, \rho(\,\mathrm{g/cm}^3) \tag{8}$$

(for $\overline{\nu_e}$ A is replaced with -A). Y_e is the electron fraction and ρ is the matter density. Density profiles through the Earth can be calculated using the Preliminary Earth Model (PREM) [12]. For neutrino trajectories through the Earth's crust, the density is typically of order 3 $\mathrm{gm/cm}^3$ ($Y_e \simeq 0.5$) and it can be considered as a constant. For very long baselines, however, this approximation is not sufficient and we must explicitly take account of ρ.

For the case of three neutrino families, the lepton mixing matrix U contains three mixing angles θ_{12}, θ_{13} and θ_{23} and one CP-violating phase δ [3] and can be built from three rotation matrices on different subspaces:

$$U = U_{23}(\theta_{23})\, U_{13}(\theta_{13}, \delta)\, U_{12}(\theta_{12}) \tag{9}$$

leading to its standard parameterization [4]:

$$U = \begin{pmatrix} c_{12}\, c_{13} & s_{12}\, c_{13} & s_{13}\, e^{-i\delta} \\ -s_{12}\, c_{23} - c_{12}\, s_{23}\, s_{13}\, e^{i\delta} & c_{12}\, c_{23} - s_{12}\, s_{23}\, s_{13}\, e^{i\delta} & s_{23}\, c_{13} \\ s_{12}\, s_{23} - c_{12}\, c_{23}\, s_{13}\, e^{i\delta} & -c_{12}\, s_{23} - s_{12}\, c_{23}\, s_{13}\, e^{i\delta} & c_{23}\, c_{13} \end{pmatrix}, \tag{10}$$

[2] The constant term $p + \frac{m_1^2}{2E_\nu}$ appears in every diagonal entries of the evolution operator and it does not affect the oscillation probabilities.

[3] In the case of Majorana neutrinos there are two additional phases but they do not affect the flavor transitions.

[4] Notice that the parameterizations differing in the position of the δ phase are physically equivalent.

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. Assuming that the ν_3 is the neutrino eigenstate that is separated from the other two, the sign of Δm^2_{13} can be either positive or negative, corresponding to the case where ν_3 is either above or below, respectively, the other two mass eigenstates. The magnitude of Δm^2_{13} determines the oscillation length of atmospheric neutrinos, while the magnitude of Δm^2_{12} determines the oscillation length of solar neutrinos, and thus $|\Delta m^2_{12}| \ll |\Delta m^2_{13}|$ (see Tab.1).

The oscillation probabilities between various flavor states can be obtained from the general expression in eq.(4) and are generally quite cumbersome. In the approximation that we neglect oscillations driven by the small Δm^2_{12} scale, only the CP-even part of the transition probabilities survives and their expression is relatively simple. As examples we show $P(\nu_e \to \nu_e)$, which is especially useful for extracting the value of the reactor angle and $P(\nu_\mu \to \nu_\tau)$, from which a good knowledge of the atmospheric mass and mixing angle can be obtained:

$$\begin{aligned} P(\nu_e \to \nu_e) &\simeq 1 - 4|U_{e3}|^2(1-|U_{e3}|^2)\sin^2\left(\tfrac{\Delta m^2_{atm}L}{4E}\right) \\ &= 1 - \sin^2(2\theta_{13})\sin^2\left(\tfrac{\Delta m^2_{atm}L}{4E}\right), \end{aligned}$$

$$\begin{aligned} P(\nu_\mu \to \nu_\tau) &\simeq 4|U_{\mu 3}|^2|U_{\tau 3}|^2\sin^2\left(\tfrac{\Delta m^2_{atm}L}{4E}\right) \\ &= \sin^2(2\theta_{23})\cos^4(\theta_{13})\sin^2\left(\tfrac{\Delta m^2_{atm}L}{4E}\right). \end{aligned} \tag{11}$$

The CP-even transition probabilities for antineutrinos can be obtained from the first relation in eq.(4).

The evaluation of the CP-odd part of the transition probabilities is a little bit more complicated due to the combined presence of two mass differences. However it is possible to obtain very simple expressions expanding the transition probabilities to second order in the small parameters θ_{13}, $\Delta m^2_{12}/\Delta m^2_{23}$ and $\Delta m^2_{12}L/2E$ [13], [14]:

$$P_{\not{CP}}(\nu_e \to \nu_\mu) = -P_{\not{CP}}(\nu_e \to \nu_\tau) = -P_{\not{CP}}(\nu_\mu \to \nu_\tau) =$$

$$\tilde{J}\,\frac{\Delta m^2_{12}L}{4E}\,\sin\delta\,\sin^2\left(\frac{\Delta m^2_{13}L}{4E}\right) \tag{12}$$

where $\tilde{J} = c_{13}\,\sin 2\theta_{12}\,\sin 2\theta_{23}\,\sin 2\theta_{13}$ (see eq.(4) for antineutrinos).

The interaction of neutrinos with matter is easily described by means of eq.(7); in the approximation where we neglect oscillations driven by the small Δm^2_{12} scale, the evolution equations are:

$$i\frac{d}{dt}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix} = \frac{\Delta m^2}{2E_\nu}\begin{pmatrix} \frac{2E_\nu}{\Delta m^2}A + |U_{e3}|^2 & U_{e3}U^*_{\mu 3} & U_{e3}U^*_{\tau 3} \\ U^*_{e3}U_{\mu 3} & |U_{\mu 3}|^2 & U_{\mu 3}U^*_{\tau 3} \\ U^*_{e3}U_{\tau 3} & U^*_{\mu 3}U_{\tau 3} & |U_{\tau 3}|^2 \end{pmatrix}\begin{pmatrix}\nu_e\\ \nu_\mu\\ \nu_\tau\end{pmatrix}. \tag{13}$$

The structure of the mixing matrix assumes a simple form for matter of constant density; in that case, the flavor eigenstates are related to the mass eigenstates ν^m_j by

$$\nu_\alpha = \sum U^m_{\alpha j}|\nu^m_j\rangle, \tag{14}$$

and

$$U^m = \begin{pmatrix} 0 & c^m_{13} & s^m_{13} \\ -c_{23} & -s^m_{13}s_{23} & c^m_{13}s_{23} \\ s_{23} & -s^m_{13}c_{23} & c^m_{13}c_{23} \end{pmatrix} \tag{15}$$

The resonance condition is obtained by looking at the relation [5]:

$$\sin^2 2\theta^m_{13} = \sin^2 2\theta_{13} / \left[\left(\frac{2E_\nu\, A}{\Delta m^2} - \cos 2\theta_{13} \right)^2 + \sin^2 2\theta_{13} \right]. \tag{16}$$

Then there is an enhancement when

$$A = \frac{\Delta m^2}{2E_\nu} \cos 2\theta_{13} \tag{17}$$

or equivalently

$$E_\nu \approx 15\,\text{GeV} \left(\frac{\Delta m^2}{3 \times 10^{-3}\,\text{eV}^2} \right) \left(\frac{1.5\,\text{g/cm}^3}{\rho Y_e} \right) \cos 2\theta_{13}\,. \tag{18}$$

The resonance occurs only for positive Δm^2 for neutrinos and only for negative Δm^2 for anti-neutrinos. For negative Δm^2 the oscillation amplitude in eq.(16) is smaller than the vacuum oscillation amplitude. Thus the matter effects give us a way in principle to determine the sign of Δm^2. It is important to stress that matter effects for long baseline experiments become important for path-lengths greater than 2000 Km.

3 Anomalies in neutrino oscillation

In the last years a certain number of hints have been collected in neutrino oscillation experiments which pointed towards the existence of sterile neutrinos [15], that is neutrinos with no weak interactions. The LSND results [16] published data showing candidate events that are consistent with $\bar{\nu}_\mu \to \bar{\nu}_e$ oscillations at very short distance. Data could be explained as if three almost degenerate (mainly active) neutrinos, accounting for the solar and atmospheric oscillations, were separated from the fourth (mainly sterile) one by the large LSND mass difference, Δm^2_{LSND}, a situation called *3+1 scheme*, see Fig.(1). The other possibility, that is two almost degenerate neutrino pairs, accounting respectively for the solar and atmospheric oscillations, separated by the LSND mass gap [17], is already ruled out [18].

For $n = 4$, the PMNS matrix contains six independent rotation angles θ_{ij} and three (if neutrinos are Dirac fermions) or six (if neutrinos are Majorana fermions) phases δ_i. This large parameter space is actually reduced to a smaller subspace whenever some of the mass differences become negligible. Consider for example the measured hierarchy in the mass differences,

$$\Delta m^2_{sol} \ll \Delta m^2_{atm} \ll \Delta m^2_{LSND}\,, \tag{19}$$

and define

$$\Delta_{ij} = \frac{\Delta m^2_{ij} L}{4E_\nu}\,; \tag{20}$$

then, at short distance of $L = O(1)$ Km, for neutrinos up to $O(10)$ GeV, the "one-mass dominance" sets in,

$$\begin{aligned} \Delta_{sol}\,,\ \Delta_{atm} &\ll 1\,, \\ \Delta_{LSND} &= O(1) \end{aligned} \tag{21}$$

[5] The case $\theta^m_{13} = \theta_{13}$ is recovered in the limit A $\to$ 0.

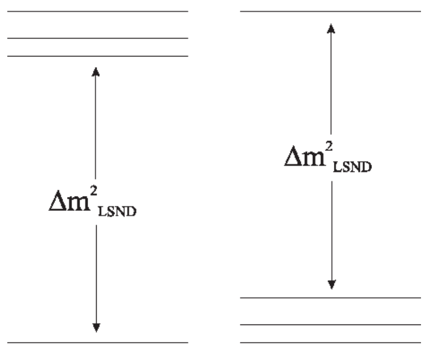

Figure 1: *Different types mass spectrum in the 3+1 scenario.*

and the whole three-dimensional subspace $(1-2-3)$ is irrelevant for short-baseline oscillation experiments; the physical parameter space now contains just three rotation angles and no phases. This means that the (otherwise) complicated neutrino transitions can be approximated by formulae depending on only one relevant mass difference and mixing angle, in the form:

$$P(\nu_\alpha \to \nu_\beta) = \delta_{\alpha\beta} - (2\delta_{\alpha\beta} - 1)\sin^2 2\theta_{\alpha\beta} \sin^2\left(\frac{\Delta m^2_{ij} L}{4E_\nu}\right), \tag{22}$$

with the meaning that

$$\begin{aligned} \sin^2 2\theta_{e\mu} &= 4|U_{e4}|^2|U_{\mu 4}|^2 && \text{for } \nu_\mu \to \nu_e \text{ appearance} \\ \sin^2 2\theta_{ee} &= 4|U_{e4}|^2(1-|U_{e4}|^2 && \text{for } \nu_e \text{ disappearance} \\ \sin^2 2\theta_{\mu\mu} &= 4|U_{\mu 4}|^2(1-|U_{\mu 4}|^2) && \text{for } \nu_\mu \text{ disappearance.} \end{aligned} \tag{23}$$

Notice that when considering CP-violating phenomena at least two mass differences should be taken into account (one neglects the solar mass difference and considers the atmospheric one as a perturbation): this is called "two-mass dominance" approximation. Regardless of the scheme, the parameter space contains 5 angles and 2 phases.

Anomalies in neutrino oscillation data belong to the following cathegories:

- the LSND results [16] provide the only positive signature of oscillations in accelerator experiments. The data showed candidate events consistent with $\overline{\nu}_\mu \to \overline{\nu}_e$ oscillations and obtained further supporting evidence by the signal in the $\nu_\mu \to \nu_e$ channel. Updated results including the runs till 1998 fixed the total fitted excess at $87.9 \pm 22.4 \pm 6$ events, corresponding to an oscillation probability of $(2.64 \pm 0.67 \pm 0.45) \times 10^{-3}$ or, in terms of oscillation parameters, to $\Delta m^2_{LSND} = 1.2$ eV^2 and $\sin^2 2\theta = 0.003$;
- a combined analysis of $\nu_\mu \to \nu_e$ together with $\overline{\nu}_\mu \to \overline{\nu}_e$ data has been published by the MiniBooNE experiment [19]; they observed an excess of events over the expected backgrounds in the low energy region part of the spectrum, below 500 MeV. The allowed region from MiniBooNE anti-neutrino data partially overlaps with the region preferred by the LSND data [20];

- anomalies in Gallium experiments (SAGE [21] and GALLEX [22]): they measured an electron neutrino flux from the Sun smaller than expected [23, 24] (ν_e disappearance experiments);
- anomalies due to new computations of reactor neutrino fluxes [25, 26]: fluxes from reactor neutrinos are $\sim$ 3.5% larger than in the past [27–30], so that experiments with L$\leq$ 100 m show deficit of neutrinos (short-baseline ν_e disappearance experiments, like Bugey [31], Rovno [32], Palo Verde [33], DoubleChooz [34]...).

In addition there are *null results* experiments, which gave no signal. Among them: ν_μ disappearance (CDHS [35], Super Kamiokande [36], MINOS [37]) and ν_e appearance (KARMEN [38], NOMAD [39], ICARUS [40], OPERA [41]).

The previous data can be analyzed in the 3+1 scheme and adopting the two-flavor probabilities given in eq.(22). Global fits of ν_e *appearance* data are consistent among themselves and give a best fit point in $(\sin^2 2\theta_{e\mu}, \Delta m^2_{41}) = (0.013, 0.42\ eV^2)$ [42]; also the ν_e *disappearance* data are not in contradiction and point toward a best fit in $(\sin^2 2\theta_{ee}, \Delta m^2_{41}) = (0.09, 1.78\ eV^2)$. On the other hand, experiments probing $\overline{\nu}_\mu$ disappearance have not reported any hints for a positive signal and this gives strong constraints on the matrix element $|U_{\mu 4}|$, that is on $\sin^2 2\theta_{\mu\mu}$. Since the three mixing angles entering eq.(23) are related by

$$\sin^2 2\theta_{e\mu} \sim \frac{1}{4} \sin^2 2\theta_{ee} \sin^2 2\theta_{\mu\mu}\,, \tag{24}$$

one would have expected a quadratic suppression on the appearance amplitude, in contrast with the obtained best fit value $\sin^2 2\theta_{e\mu} = 0.013$. Roughly speaking, this is the source of the well-known tension between appearance and disappearance experiments. To quantify the consistency of different parts of the global data, one can use the so-called parameter goodness of fit test; the authors of Ref. [42] found a value around 10^{-4}, which strongly indicates the poor agreement of the two sets of data.

Notice that the 3+1 fit is much improved if the low energy MiniBooNE data are not included [43]. In fact, this subset of the data is incompatible with neutrino oscillations since they require a small value of Δm^2_{14} and a large value of $\sin^2 2\theta_{e\mu}$ well beyond what is allowed by the data of other experiments. Removing the the low energy excess of MiniBooNE from the fit, the authors of Ref. [43] found a parameter goodness of fit as large as 9%.

In conclusion, the situation is at present quite confuse and additional experimental effort is needed to establish the existence of sterile neutrinos.

4 Models of masses and mixing

4.1 Models based on discrete symmetries

Looking at the results of Tab.1, one is still tempted to recognize some special mixing patterns as good first approximations to describe the data, the most famous ones being the Tri-Bimaximal (TB [44]), the Golden Ratio (GR [45]) and the Bi-Maximal (BM) mixing. The corresponding mixing matrices all have:

$$\sin^2\theta_{23} = 1/2, \qquad \sin^2\theta_{13} = 0 \tag{25}$$

and differ by the value of the solar angle $\sin^2\theta_{12}$, which is:

$$\sin^2\theta_{12} = 1/3 \text{ for TB}, \qquad \sin^2\theta_{12} = \frac{2}{5+\sqrt{5}} \sim 0.276 \text{ for GR}, \qquad \sin^2\theta_{12} = \frac{1}{2} \text{ for BM}\,. \tag{26}$$

Being a leading order approximations (LO), all previous patterns need corrections (for example, from the diagonalization of charged leptons) to describe the current mixing angles. In particular, the relatively large value of the reactor angle requires sizable corrections of the order of the Cabibbo angle λ_C, for all three patterns; on the other hand, the deviations from the LO values of $\sin^2\theta_{12}$ must be small enough in the TB and GR cases but as large as λ_C for the BM pattern. Finally, corrections not too much larger than λ_C^2 can be tolerated by $\sin^2\theta_{23}$. Since the corresponding mixing matrices have the form of rotations with special angles, discrete flavor groups naturally emerge as good candidates. The most studied groups have been the permutation groups of four object, S_4 and A_4, see Ref. [46] for an exhaustive review and [47] for examples involving A_4. The important point for model building is that these symmetries must be broken by suitable scalar fields φ that take a vacuum expectation value (vev) at large scale Λ, so they generally provide a new adimensional parameter $\xi = \langle\varphi\rangle/\Lambda$. The breaking must preserve different subgroups in the charged lepton and neutrino sectors, otherwise the neutrino mixing matrix would be the identity matrix and no mixing will be generated. The desired directions in flavor space are generally difficult to achieve, so consistent models are those where the vevs of the scalar fields can be naturally obtained from the minimization of the scalar potential.

Beside the models based on TB, one can consider models where BM mixing holds in the neutrino sector at LO and the relatively large corrective terms for θ_{12} and θ_{13}, of $\mathcal{O}(\lambda_C)$, arise from the diagonalization of charged lepton masses; the atmospheric angle, however, should deviate from maximal mixing by quantities not much larger than $\mathcal{O}(\lambda_C^2)$. Explicit models of this type based on the group S_4 have been developed in Ref. [48].

For the BM mixing, the mixing matrix has the form

$$U_{BM} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{27}$$

corresponding to the following mass matrix:

$$m_{\nu BM} = \begin{pmatrix} x & y & y \\ y & z & x-z \\ y & x-z & z \end{pmatrix}, \tag{28}$$

where x,y and z are three complex numbers. One can consider the possibility that BM is the mixing in the neutrino sector and that the rather large corrective terms to θ_{12} and θ_{13} arise from the diagonalization of the charged lepton mass matrix, as obtained in models based on the discrete symmetry S_4 [48,49]. This idea is in agreement with the well-known empirical quark-lepton complementarity relation [50]- [51], $\theta_{12} + \theta_C \sim \pi/4$, where θ_C is the Cabibbo angle or, to be less optimistic, with the "weak" complementarity relation $\theta_{12} + \mathcal{O}(\theta_C) \sim \pi/4$. In addition, the measured value of θ_{13} is itself of order θ_C: $\theta_{13} \sim \theta_C/\sqrt{2}$.

In the following, two examples of GUT models based on BM will be considered [52]: one is based on $SU(5)$ [49] and realizes the program of imposing the BM structure in the neutrino sector and then correcting it by terms arising from the diagonalization of charged lepton masses. The other is an $SO(10)$ model based on Type-II see-saw [53], where the origin of BM before diagonalization of charged leptons is left unspecified.

In the first case we deal with a variant of the SUSY $SU(5)$ model in 4+1 dimensions with a flavor symmetry $S_4 \otimes Z_3 \otimes U(1)_R \otimes U(1)_{FN}$ [48, 49], where $U(1)_R$ implements the R-symmetry while $U(1)_{FN}$ is a

Field	F	T_1	T_2	T_3	H_5	$H_{\bar{5}}$	φ_ν	ξ_ν	φ_ℓ	χ_ℓ	θ	θ'	φ_ν^0	ξ_ν^0	ψ_ℓ^0	χ_ℓ^0
SU(5)	$\bar{5}$	10	10	10	5	$\bar{5}$	1	1	1	1	1	1	1	1	1	1
S_4	3_1	1	1	1	1	1	3_1	1	3_1	3_2	1	1	3_1	1	2	3_2
Z_3	ω	ω	1	ω^2	ω^2	ω^2	1	1	ω	ω	1	ω	1	1	ω	ω
$U(1)_R$	1	1	1	1	0	0	0	0	0	0	0	0	2	2	2	2
$U(1)_{FN}$	0	2	1	0	0	0	0	0	0	0	-1	-1	0	0	0	0
	br	bu	bu	br	bu	bu	br	br	br	br	br	br	br	br	br	br

Table 2: *Matter assignment of the model. The symbol* br(bu) *indicates that the corresponding fields live on the brane (bulk).*

Froggatt-Nielsen (FN) symmetry [54] that induces the hierarchies of fermion masses and mixings. The particle assignments are displayed in Tab.2. The first two generation tenplets T_1 and T_2 and the Higgs H_5 and $H_{\bar{5}}$ are in the bulk while all the other ones are on the brane at $y = 0$; this introduces some extra hierarchy for some of the couplings [55]- [58]. At leading order (LO) the S_4 symmetry is broken down to suitable different subgroups in the charged lepton sector and in the neutrino sector by the VEV's of the flavons φ_ν, ξ_ν, φ_ℓ and χ_ℓ (whose proper alignment is implemented in a natural way by the driving fields φ_ν^0, ξ_ν^0, ψ_ℓ^0, χ_ℓ^0). The VEVs of the θ and θ' fields break the FN symmetry. As a result, at LO the charged lepton masses are diagonal and exact BM is realized for neutrinos. Corrections to diagonal charged leptons and to exact BM are induced by vertices of higher dimension in the Lagrangian, suppressed by powers of a large scale Λ. We adopt the definitions:

$$\frac{v_{\varphi_\ell}}{\Lambda} \sim \frac{v_\chi}{\Lambda} \sim \frac{v_{\varphi_\nu}}{\Lambda} \sim \frac{v_\xi}{\Lambda} \sim \frac{\langle\theta\rangle}{\Lambda} \sim \frac{\langle\theta'\rangle}{\Lambda} \sim s \equiv \lambda_C \,, \tag{29}$$

where $s = \dfrac{1}{\sqrt{\pi R \Lambda}}$ is the volume suppression factor. It turns out that this simple choice leads to a good description of masses and mixings.

For the charged lepton masses, the matter assignment of the model gives rise to the following mass matrix:

$$m_e \sim \begin{pmatrix} a_{11}\lambda^5 & a_{21}\lambda^4 & a_{31}\lambda^2 \\ a_{12}\lambda^4 & -c\lambda^3 & ...... \\ a_{13}\lambda^4 & c\lambda^3 & a_{33}\lambda \end{pmatrix} \lambda, \tag{30}$$

where all matrix elements are multiplied by generic coefficients of $\mathcal{O}(1)$. The corresponding lepton rotation is given by:

$$U_\ell \sim \begin{pmatrix} 1 & u_{12}\lambda & u_{13}\lambda \\ -u_{12}^*\lambda & 1 & 0 \\ -u_{13}^*\lambda & -u_{12}^* u_{13}^*\lambda^2 & 1 \end{pmatrix}, \tag{31}$$

(u_{ij} again of $\mathcal{O}(1)$) so that $\theta_{23}^\ell = 0$ in this approximation.

The neutrino sector of the model is unchanged with respect to Ref. [49]. At LO, the mass matrix of eq.(28) is obtained from the Weinberg operator, so the results for the mixing angles are easily derived:

$$\sin\theta_{13} \;=\; \frac{1}{\sqrt{2}}|u_{12} - u_{13}|\lambda \qquad \sin^2\theta_{12} = \frac{1}{2} - \frac{1}{\sqrt{2}}\, Re(u_{12} + u_{13})\lambda \qquad \sin^2\theta_{23} = \frac{1}{2} + \mathcal{O}(\lambda^2)\,.$$

We see that, with $\lambda \sim \lambda_C$, the model realizes the "weak" complementarity relation and the experimental fact that $\sin\theta_{13}$ is of the same order than the shift of $\sin^2\theta_{12}$ from the BM value of $1/2$, both of order λ_C.

The second model presented here is based on the $SO(10)$ gauge group. In $SO(10)$ the main added difficulty with respect to $SU(5)$ is that one generation of fermion belongs to the 16-dimensional representation, so that one cannot take advantage of the properties of the $SU(5)$-singlet right-handed neutrinos. A possible strategy to separate charged fermions and neutrinos is to assume the dominance of type-II see-saw with respect to the more usual type-I see-saw. In models of this type, the neutrino mass formula becomes

$$\mathcal{M}_\nu \sim f v_L , \tag{32}$$

where v_L is the vev of the $B - L = 2$ triplet in the $\overline{\mathbf{126}}$ Higgs field and f is the Yukawa coupling matrix of the $\mathbf{16}$ with the same $\overline{\mathbf{126}}$.

For generic eigenvalues m_i, the most general matrix that is diagonalized by the BM unitary transformation is given by:

$$f = U^*_{BM}\mathrm{diag}(m_1, m_2, m_3)U^\dagger_{BM} , \tag{33}$$

where U_{BM} is the BM mixing matrix given in eq.(27). However, a similar transformation can also be used with U_{BM} replaced by U_{TB}; as a result, the matrices f obtained with this two different approaches are related by a change of the charged lepton basis induced by a unitary matrix. As one could decide to work in a basis where the matrix f is diagonalized by the TB matrix or by BM matrix, the result of a fit performed in one basis should lead to the same χ^2 than the fit in other basis, so the χ^2 cannot decide whether TB or BM is a better starting point. Then we need another "variable" to compare whether the data prefer to start from TB or BM. One possibility is to measure the amount of fine-tuning needed to fit a set of data; to this aim, a parameter d_{FT} was introduced in Ref. [53]:

$$d_{FT} = \sum \mid \frac{par_i}{err_i} \mid, \tag{34}$$

where err_i is the "error" of a given parameter par_i defined as the shift from the best fit value that changes the χ^2 by one unit, with all other parameters fixed at their best fit values.

A study of the fine tuning parameter when the fit is repeated with the same data except for $\sin^2\theta_{13}$, which is moved from small to large, shows that the fine tuning increases (decreases) with $\sin\theta_{13}$ for TB (BM), as shown in Fig.(2).

A closer look at the figure reveals that both BM and TB scenarios are compatible with the data for similar values of the fine tuning parameter, especially for relatively large θ_{13}. We have also observed that high d_{FT} values are predominantly driven by the smallness of the electron mass combined with its extraordinary measurement precision.

4.2 Models based on abelian $U(1)$

The relatively large value of θ_{13} and the fact that θ_{23} is not maximal both point to the direction of models based on Anarchy [59,60], that is the assumption that no special symmetry is needed in the leptonic sector, and that the values of neutrino masses and mixing are reproduced by chance. Anarchy can be formulated in a $U(1)$ context a la Froggatt-Nielsen [54]: a mass term is allowed at the renormalisable level only if the $U(1)_{FN}$ charges add to zero. Breaking the $U(1)_{FN}$ symmetry spontaneously by the vevs v_f of a number of flavon fields with non-vanishing charge allows to rescue the forbidden vertex, although suppressed by powers of the small parameters $\lambda = v_f/M$, with M a large mass scale. Since these invariant mass terms

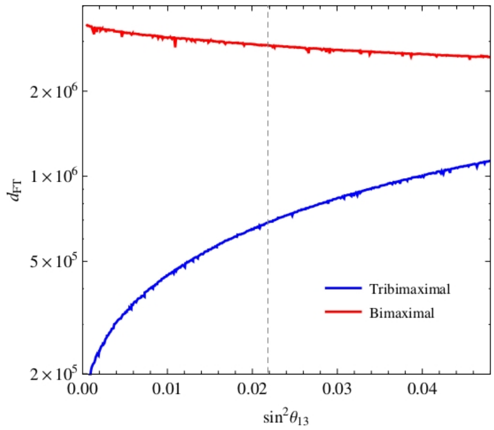

Figure 2: *In the $SO(10)$ model the fine tuning parameter d_{FT} increases (decreases) with $\sin^2\theta_{13}$ in the TB (BM) cases. For the physical value $\sin^2\theta_{13} \sim 0.022$ it is about 4 times larger in the BM case.*

appear with arbitrary coefficients of order 1, typically the number of parameters exceeds the number of observable quantities and make this kind of models less predictive than the ones based on non-abelian discrete symmetries.

Opposite to Anarchy, generic $U(1)$ models are characterized by well-defined hierarchies of the neutrino mass matrix elements and are often referred to as hierarchical models. The authors of Ref. [61] have performed an updated analysis of the performance of anarchical versus hierarchical models in the $SU(5) \otimes U(1)_{FN}$ context, which also allows to implement a parallel treatment of quarks and leptons. Among the different charge assignments of the 10, $\overline{5}$ and the $SU(5)$ singlet, we focus here on two different realizations:

- Anarchy (A): 10=(3,2,0), $\overline{5}$=(0,0,0) 1=(0,0,0)

$$m_\ell = \begin{pmatrix} \lambda^3 & \lambda^3 & \lambda^3 \\ \lambda^2 & \lambda^2 & \lambda^2 \\ 1 & 1 & 1 \end{pmatrix}, \qquad m_\nu = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{35}$$

- Hierarchy (H): 10=(5,3,0), $\overline{5}$=(2,1,0) 1=(2,1,0)

$$m_\ell = \begin{pmatrix} \lambda^7 & \lambda^6 & \lambda^5 \\ \lambda^5 & \lambda^4 & \lambda^3 \\ \lambda^2 & \lambda & 1 \end{pmatrix}, \qquad m_\nu = \begin{pmatrix} \lambda^4 & \lambda^3 & \lambda^2 \\ \lambda^3 & \lambda^2 & \lambda \\ \lambda^2 & \lambda & 1 \end{pmatrix}. \tag{36}$$

The values of the neutrino observables were computed extracting the modulus (argument) of the complex random coefficients in the interval $[0.5, 2]$ ($[0, 2\pi]$) with a flat distribution. In order to ensure a reasonable hierarchy for charged fermions, the values of λ are different in the two cases: $\lambda = 0.2$ for A, $\lambda = 0.4$ for H. The results of such a scan can be summarized as follows. Since the problem with Anarchy is that all mixing angles should be large and of the same order of magnitude, it is quite difficult to reproduce θ_{13} of the order of the Cabibbo angle. In addition, the smallness of solar-to-atmospheric mass hierarchy r is not easily reproduced, being generically one order of magnitude larger than expected. On the other hand, in the H model one can reproduce the correct size for r and $\sin^2\theta_{13}$, thus making this option preferable over Anarchy. These results have been confirmed by a more recent analysis in [63].

5 Summary and Outlook

Neutrino physics deals with fundamental issues still of great importance, as the origin of masses and mixing and the possible existence of sterile neutrino species. In the domain of model building, an intense work to interpret the new data (especially after the recent measurement of θ_{13}) has span a wide range of possibilities: beyond the ones illustrated here, one should also mention the use of larger symmetries that already at LO lead to non vanishing θ_{13} and non maximal θ_{23} [64] to models where the flavor group and a generalized CP transformation are combined in a non trivial way [65,66]. In spite of this huge effort, not a single model has been built which contains all features of the neutrino experimental data without invoking some fine-tuning in the model parameters or ad-hoc constructions. Thus, more work must be devoted to this field to clarify the origin of the present experimental data.

Acknowledgments

I am very grateful to the Organizers of the QFTHEP'2015 workshop for inviting me to give this review talk and for the beautiful and stimulating atmosphere during my stay in Samara.

References

[1] **T2K** Collaboration, K. Abe *et. al.*, Phys. Rev. Lett. **107** (2011) 041801, arXiv:1106.2822; arXiv:1106.2822.

[2] **MINOS** Collaboration, P. Adamson *et. al.*, Phys. Rev. Lett. **107** (2011) 181802, arXiv:1108.0015.

[3] Y. Abe *et al.* [Double Chooz Collaboration], Phys. Rev. Lett. **108**, 131801 (2012) doi:10.1103/PhysRevLett.108.131801 [arXiv:1112.6353 [hep-ex]].

[4] J. K. Ahn *et al.* [RENO Collaboration], Phys. Rev. Lett. **108**, 191802 (2012) doi:10.1103/PhysRevLett.108.191802 [arXiv:1204.0626 [hep-ex]].

[5] F. P. An *et al.* [Daya Bay Collaboration], Phys. Rev. Lett. **108**, 171803 (2012) doi:10.1103/PhysRevLett.108.171803 [arXiv:1203.1669 [hep-ex]].

[6] F. Capozzi, G. L. Fogli, E. Lisi, A. Marrone, D. Montanino and A. Palazzo, Phys. Rev. D **89**, 093018 (2014) doi:10.1103/PhysRevD.89.093018 [arXiv:1312.2878 [hep-ph]].

[7] M. C. Gonzalez-Garcia, M. Maltoni and T. Schwetz, arXiv:1512.06856 [hep-ph].

[8] D. V. Forero, M. Tortola and J. W. F. Valle, Phys. Rev. D **90**, no. 9, 093006 (2014) doi:10.1103/PhysRevD.90.093006 [arXiv:1405.7540 [hep-ph]].

[9] B. Pontecorvo, Sov. Phys. JETP **6**, 429 (1957) [Zh. Eksp. Teor. Fiz. **33**, 549 (1957)];
Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962). doi:10.1143/PTP.28.870.

[10] L. Wolfenstein, Phys. Rev. D **17**, 2369 (1978).

[11] M. C. Gonzalez-Garcia and Y. Nir, Rev. Mod. Phys. **75**, 345 (2003) [hep-ph/0202058].

[12] A. Dziewonski, Earth Structure, Global, in "The Encyclopedia of Solid Earth Geophysics", ed. by D.E. James, (Van Nostrand Reinhold, New York, 1989);
R. Gandhi, C. Quigg, M. Hall Reno, and I. Sarcevic, Astroparticle Physics **5**, 81 (1996).

[13] A. Cervera, A. Donini, M. B. Gavela, J. J. Gomez Cadenas, P. Hernandez, O. Mena and S. Rigolin, Nucl. Phys. B **579**, 17 (2000) [Nucl. Phys. B **593**, 731 (2001)] [hep-ph/0002108].

[14] A. Donini, D. Meloni and P. Migliozzi, Nucl. Phys. B **646**, 321 (2002) [hep-ph/0206034].

[15] K. Abazajian *et. al.*, arXiv:1204.5379.

[16] C. Athanassopoulos *et al.* [LSND Collaboration], Phys. Rev. Lett. **75**, 2650 (1995) [nucl-ex/9504002].

[17] A. Donini and D. Meloni, Eur. Phys. J. C **22**, 179 (2001) doi:10.1007/s100520100777 [hep-ph/0105089].

[18] M. Maltoni, T. Schwetz, M. A. Tortola and J. W. F. Valle, Nucl. Phys. B **643**, 321 (2002) doi:10.1016/S0550-3213(02)00747-2 [hep-ph/0207157].

[19] A. A. Aguilar-Arevalo *et al.* [MiniBooNE Collaboration], Phys. Rev. Lett. **110**, 161801 (2013) doi:10.1103/PhysRevLett.110.161801 [arXiv:1207.4809 [hep-ex], arXiv:1303.2588 [hep-ex]]; Phys. Rev. Lett. **110**, 161801 (2013) doi:10.1103/PhysRevLett.110.161801 [arXiv:1207.4809 [hep-ex], arXiv:1303.2588 [hep-ex]].

[20] C. Athanassopoulos *et al.* [LSND Collaboration], Phys. Rev. Lett. **75**, 2650 (1995) doi:10.1103/PhysRevLett.75.2650 [nucl-ex/9504002]; Phys. Rev. Lett. **77**, 3082 (1996) doi:10.1103/PhysRevLett.77.3082 [nucl-ex/9605003]; Phys. Rev. Lett. **81**, 1774 (1998) doi:10.1103/PhysRevLett.81.1774 [nucl-ex/9709006].

[21] J. N. Abdurashitov *et al.* [SAGE Collaboration], Phys. Rev. C **59**, 2246 (1999) [hep-ph/9803418].

[22] W. Hampel *et al.* [GALLEX Collaboration], Phys. Lett. B **420**, 114 (1998).

[23] M. A. Acero, C. Giunti and M. Laveder, Phys. Rev. D **78**, 073009 (2008) [arXiv:0711.4222 [hep-ph]].

[24] C. Giunti and M. Laveder, Phys. Rev. C **83**, 065504 (2011) [arXiv:1006.3244 [hep-ph]].

[25] G. Mention, M. Fechner, T. Lasserre, T. A. Mueller, D. Lhuillier, M. Cribier and A. Letourneau, Phys. Rev. D **83**, 073006 (2011) doi:10.1103/PhysRevD.83.073006 [arXiv:1101.2755 [hep-ex]].

[26] P. Huber, Phys. Rev. C **84**, 024617 (2011) [Phys. Rev. C **85**, 029901 (2012)] doi:10.1103/PhysRevC.85.029901, 10.1103/PhysRevC.84.024617 [arXiv:1106.0687 [hep-ph]].

[27] P. Vogel, G. K. Schenter, F. M. Mann and R. E. Schenter, Phys. Rev. C **24**, 1543 (1981).

[28] F. Von Feilitzsch, A. A. Hahn and K. Schreckenbach, Phys. Lett. B **118**, 162 (1982).

[29] K. Schreckenbach, G. Colvin, W. Gelletly and F. Von Feilitzsch, Phys. Lett. B **160**, 325 (1985).

[30] A. A. Hahn, K. Schreckenbach, G. Colvin, B. Krusche, W. Gelletly and F. Von Feilitzsch, Phys. Lett. B **218**, 365 (1989).

[31] Y. Declais *et al.*, Phys. Lett. B **338**, 383 (1994).

[32] A. A. Kuvshinnikov, L. A. Mikaelyan, S. V. Nikolaev, M. D. Skorokhvatov and A. V. Etenko, JETP Lett. **54** (1991) 253 [Yad. Fiz. **52** (1990) 472] [Pisma Zh. Eksp. Teor. Fiz. **54** (1991) 259] [Sov. J. Nucl. Phys. **52** (1990) 300].

[33] F. Boehm *et al.*, Phys. Rev. D **64** (2001) 112001 [hep-ex/0107009].

[34] Y. Abe *et al.* [Double Chooz Collaboration], Phys. Rev. D **86**, 052008 (2012) [arXiv:1207.6632 [hep-ex]].

[35] F. Dydak *et al.*, Phys. Lett. B **134**, 281 (1984).

[36] S. M. Bilenky, C. Giunti, W. Grimus and T. Schwetz, Phys. Rev. D **60** (1999) 073007 [hep-ph/9903454].

[37] P. Adamson *et al.* [MINOS Collaboration], Phys. Rev. D **81** (2010) 052004 [arXiv:1001.0336 [hep-ex]].

[38] B. Armbruster *et al.* [KARMEN Collaboration], Phys. Rev. D **65**, 112001 (2002) [hep-ex/0203021].

[39] P. Astier *et al.* [NOMAD Collaboration], Phys. Lett. B **570**, 19 (2003) [hep-ex/0306037].

[40] M. Antonello *et al.*, Eur. Phys. J. C **73**, no. 3, 2345 (2013) doi:10.1140/epjc/s10052-013-2345-6 [arXiv:1209.0122 [hep-ex]].

[41] N. Agafonova *et al.* [OPERA Collaboration], JHEP **1307**, 004 (2013) [JHEP **1307**, 085 (2013)] doi:10.1007/JHEP07(2013)004, 10.1007/JHEP07(2013)085 [arXiv:1303.3953 [hep-ex]].

[42] J. Kopp, P. A. N. Machado, M. Maltoni and T. Schwetz, JHEP **1305**, 050 (2013) doi:10.1007/JHEP05(2013)050 [arXiv:1303.3011 [hep-ph]].

[43] C. Giunti, M. Laveder, Y. F. Li and H. W. Long, Phys. Rev. D **88**, 073008 (2013) doi:10.1103/PhysRevD.88.073008 [arXiv:1308.5288 [hep-ph]].

[44] P. F. Harrison, D. H. Perkins, and W. G. Scott, Phys. Lett. **B530** (2002) 167, arXiv:hep-ph/0202074; P. F. Harrison and W. G. Scott, , Phys. Lett. **B535** (2002) 163, arXiv:hep-ph/0203209; Z.-z. Xing, Phys. Lett. **B533** (2002) 85, arXiv:hep-ph/0204049; P. F. Harrison and W. G. Scott, Phys. Lett. **B547** (2002) 219, arXiv:hep-ph/0210197; Phys. Lett. **B557**(2003) 76, arXiv:hep-ph/0302025.

[45] Y. Kajiyama, M. Raidal, and A. Strumia, Phys. Rev. **D76** (2007) 117301,arXiv:0705.4559; L. L. Everett and A. J. Stuart, Phys. Rev. **D79** (2009) 085005, arXiv:0812.1057; G.-J. Ding, L. L. Everett, and A. J. Stuart, Nucl. Phys. **B857** (2012) 219, arXiv:1110.1688; F. Feruglio and A. Paris, JHEP **03** (2011) 101, arXiv:1101.0393.

[46] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010) [arXiv:1002.0211 [hep-ph]].

[47] G. Altarelli and F. Feruglio, Nucl. Phys. **B741** (2006) 215, arXiv:hep-ph/0512103; G. Altarelli and D. Meloni, J. Phys. G **36**, 085005 (2009) [arXiv:0905.0620 [hep-ph]].

[48] G. Altarelli, F. Feruglio and L. Merlo, JHEP **0905**, 020 (2009) [arXiv:0903.1940 [hep-ph]].

[49] D. Meloni, JHEP **1110** (2011) 010 [arXiv:1107.0221 [hep-ph]].

[50] M. Raidal, Phys. Rev. Lett. **93**, 161801 (2004) [hep-ph/0404046].

[51] S. Antusch, S. F. King and R. N. Mohapatra, Phys. Lett. B **618**, 150 (2005) [hep-ph/0504007].

[52] G. Altarelli, P. A. N. Machado and D. Meloni, PoS CORFU **2014** (2015) 012 [arXiv:1504.05514 [hep-ph]].

[53] G. Altarelli and G. Blankenburg, JHEP **1103** (2011) 133 [arXiv:1012.2697 [hep-ph]].

[54] C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B **147** (1979) 277.

[55] E. Witten, Nucl. Phys. B **258** (1985) 75; Y. Kawamura, Prog. Theor. Phys. **105** (2001) 999 [arXiv:hep-ph/0012125]; A. E. Faraggi, Phys. Lett. B **520** (2001) 337 [arXiv:hep-ph/0107094] and references therein.

[56] L. J. Hall and Y. Nomura, Phys. Rev. D **64** (2001) 055003 [arXiv:hep-ph/0103125]; Y. Nomura, Phys. Rev. D **65** (2002) 085036 [arXiv:hep-ph/0108170]; L. J. Hall and Y. Nomura, Phys. Rev. D **66** (2002) 075004 [arXiv:hep-ph/0205067].

[57] G. Altarelli and F. Feruglio, Phys. Lett. B **511** (2001) 257 [arXiv:hep-ph/0102301]; A. Hebecker and J. March-Russell, Nucl. Phys. B **613** (2001) 3 [arXiv:hep-ph/0106166]; A. Hebecker and J. March-Russell, Phys. Lett. B **541** (2002) 338 [arXiv:hep-ph/0205143].

[58] G. Altarelli, F. Feruglio, C. Hagedorn, JHEP **0803**, 052-052 (2008). [arXiv:0802.0090 [hep-ph]].

[59] L. J. Hall, H. Murayama, and N. Weiner, Phys. Rev. Lett. **84** (2000) 2572, arXiv:hep-ph/9911341.

[60] A. de Gouvea and H. Murayama, Phys. Lett. **B573** (2003) 94, arXiv:hep-ph/0301050.

[61] G. Altarelli, F. Feruglio, I. Masina and L. Merlo, JHEP **1211**, 139 (2012) [arXiv:1207.0587 [hep-ph]].

[62] A. de Gouvea and H. Murayama, Phys. Lett. B **747**, 479 (2015) doi:10.1016/j.physletb.2015.06.028 [arXiv:1204.1249 [hep-ph]].

[63] J. Bergstrom, D. Meloni and L. Merlo, Phys. Rev. D **89** (2014) 9, 093021 [arXiv:1403.4528 [hep-ph]].

[64] R. d. A. Toorop, F. Feruglio and C. Hagedorn, Phys. Lett. B **703** (2011) 447 [arXiv:1107.3486 [hep-ph]]; Nucl. Phys. B**858**, 437 (2012) , arXiv:1112.1340; S. F. King, C. Luhn and A. J. Stuart, Nucl. Phys. B **867** (2013) 203 [arXiv:1207.5741 [hep-ph]]; C. Hagedorn and D. Meloni, Nucl. Phys. B **862** (2012) 691 [arXiv:1204.0715 [hep-ph]]; S. F. King, T. Neder and A. J. Stuart, Phys. Lett. B **726** (2013) 312 [arXiv:1305.3200 [hep-ph]];

[65] G. Ecker, W. Grimus and H. Neufeld, J. Phys. A **20** (1987) L807; Int. J. Mod. Phys. A **3** (1988) 603; W. Grimus and M. N. Rebelo, Phys. Rept. **281** (1997) 239 [hep-ph/9506272]; W. Grimus and L. Lavoura, Phys. Lett. B **579** (2004) 113 [hep-ph/0305309]; M. Holthausen, M. Lindner and M. A. Schmidt, JHEP **1304** (2013) 122 [arXiv:1211.6953 [hep-ph]]; F. Feruglio, C. Hagedorn and R. Ziegler, JHEP **1307** (2013) 027 [arXiv:1211.5560 [hep-ph]]; Eur. Phys. J. C **74** (2014) 2753 [arXiv:1303.7178 [hep-ph]].

[66] I. de Medeiros Varzielas and D. Emmanuel-Costa, Phys. Rev. D **84** (2011) 117901 [arXiv:1106.5477 [hep-ph]]; G. Bhattacharyya, I. de Medeiros Varzielas and P. Leser, Phys. Rev. Lett. **109** (2012) 241603 [arXiv:1210.0545 [hep-ph]]; M. C. Chen and K. T. Mahanthappa, Phys. Lett. B **681** (2009) 444 [arXiv:0904.1721 [hep-ph]]; L. L. Everett, T. Garon and A. J. Stuart, JHEP **1504**, 069 (2015) doi:10.1007/JHEP04(2015)069 [arXiv:1501.04336 [hep-ph]]; C. C. Li and G. J. Ding, JHEP **1505**, 100 (2015) doi:10.1007/JHEP05(2015)100 [arXiv:1503.03711 [hep-ph]]; A. Di Iura, C. Hagedorn and D. Meloni, JHEP **1508**, 037 (2015) doi:10.1007/JHEP08(2015)037 [arXiv:1503.04140 [hep-ph]].